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A comparison of second-order constitutive theories for hyperelastic materials

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Abstract

To develop a finite elastic constitutive equation for a real material, it is necessary to experimentally determine a number of material functions. This is much more difficult than determining the material constants needed for construction of a second-order constitutive equation for the same material. So it can be advantageous to use a second-order constitutive equation if it provides an adequate description of a material's mechanical behavior over the deformation regimes to be considered. Of course, the second-order expansion of a constitutive relation always provides a good description of the mechanical response for sufficiently small strains and rotations. But by neglecting terms in these expansions which are higher than second order, we can construct a St Venant–Kirchhoff-type material model which may be applied for any deformation. In this paper we investigate the circumstances under which such St Venant–Kirchhoff-type second-order constitutive equations can successfully be used in lieu of the fully nonlinear elastic constitutive equation to solve boundary value problems. Two nonlinear elastic materials are considered: the generalized Blatz–Ko and the harmonic materials. For these materials we examine the performance of constitutive equations which are second order in the displacement gradient, the Biot strain, and the Green strain. A variety of boundary value problems are solved with these second-order constitutive equations, and the resulting solutions are compared to the solutions obtained with the corresponding fully nonlinear constitutive equation. We find that both the nature of the material and the nature of the deformation have a large impact on the performance of these three second-order constitutive equations. Overall, the constitutive equation which is second order in the Biot strain provides the most accurate solutions over the largest range of strains and rotations. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Second-order elasticity; St. Venant–Kirchhoff material; Biot strain

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1. Introduction

There have been two main impediments to the solution of boundary value problems in finite elasticity: the nonlinear nature of the underlying equations; and the lack of constitutive information for specific elastic materials. Consequently, over much of the fifty-year history of nonlinear elasticity, it has been very difficult to solve any but the simplest problems. The finite element and the finite difference methods together with the accessibility of powerful computers have greatly reduced the difficulties of solving nonlinear equations, effectively removing the first impediment. But the obstacles to constructing finite elastic constitutive equations remain.

The development of realistic finite elastic constitutive equations for real materials can be a very difficult task because it requires experimental determination of a number of *material functions*. Construction of a constitutive equation to describe the mechanical behavior of complex nonlinear materials even in a limited deformation regime can be extremely challenging. This is particularly true for materials, such as biological tissues, for which only limited constitutive information can be obtained because accurate experimental testing present major technical obstacles (see, for example, Humphrey, 1995).

In contrast, the development of a linear or a second-order constitutive equation requires only that a number of *material constants* be determined, which is a much more tractable undertaking. For example, to construct a nonlinear isotropic hyperelastic constitutive equation requires that the strain energy be determined as a function of the three principal strain invariants (or an equivalent set of invariants). In contrast, a second-order approximation to this constitutive equation requires only that five constants be determined. Thus, it can be very advantageous to use a second-order constitutive equation while retaining the full geometric nonlinearity in a boundary value problem, provided the second-order constitutive relation gives an adequate description of the mechanical behavior of the material over the deformation regime that is to be modeled.

The approach of using a second-order constitutive equation for problems in which the strain may be substantial is analogous to the approach taken by St Venant (1844) and Kirchhoff (1852) in defining the nonlinear St Venant–Kirchhoff material. This material is based on the form of the isotropic constitutive relation for classical linear elasticity, with the Piola–Kirchhoff stress linear in the Green strain. Although a number of difficulties have been identified for the St Venant–Kirchhoff material (see, for example, Ciarlet, 1988), this nonlinear model has been employed often because of its extreme simplicity.

A number of second-order constitutive equations have been proposed, usually in the context of isotropic materials. The vast majority of those theories are for hyperelastic materials and use the displacement gradient as the measure of the deformation (for example, see Murnaghan, 1937; Rivlin, 1952; Toupin and Bernstein, 1961; Ogden, 1984; Lindsay, 1992; Haughton and Lindsay, 1993, 1994). Second-order theories have also been developed for Cauchy elastic materials in terms of the displacement gradient (see, for example, Sheng, 1955). Many of these constitutive equations are summarized and compared in Truesdell and Noll (1965). A substantial shortcoming of these second-order constitutive equations formulated in terms of the displacement gradient is that they do not satisfy material frame indifference, so solutions to problems with large or even moderate rotations will be inherently in error.

Frame indifferent second-order constitutive equations have also been presented: Murnaghan obtained an explicit constitutive equation for isotropic hyperelastic materials that was second-order in the Green strain[1] (1951); Ogden gave a second-order expansion of the Kirchhoff stress (1978) and a general outline

[1] Unlike the situations in classical infinitesimal elasticity and in finite elasticity, the choice of strain measure is important in the derivation of a second-order theory. (This has been clearly recognized in the past; see for example, Ogden, 1984, p. 350.) This is due to the facts that a constitutive equation which is second order in one strain measure will not be second order in a different strain measure, and the each strain measure typically contains a different power of the stretch.

of the expansion of the second Piola–Kirchhoff stress (1984), both in terms of the Green strain; and recently Hoger (1998) introduced a method that can be used to obtain a constitutive relation which is second-order in the Biot strain from a hyperelastic material with arbitrary material symmetry.

Of course all of the various second-order constitutive relations will give, essentially, the same accuracy over some suitably small range of strains and rotations, and those that are frame indifferent will provide accurate solutions for problems in which the strains are sufficiently small.

However, if these second-order relations are viewed as St Venant–Kirchhoff-type constitutive equations and used for solving problems with large strains and rotations, significant differences in the predicted solutions will emerge. In this circumstance one may ask which of the second-order constitutive equations gives the solution that best approximates the solution of the fully nonlinear constitutive theory over the largest range of deformations or strains. In addition, it would be useful to determine the range of applicability of such St Venant–Kirchhoff-type second-order constitutive equations for specific materials. These are the primary issues that are addressed in this paper. Of course, the answers will depend on the geometry of the problem, the nature of the material, and on the level of error that can be tolerated in the solution. However, we hope to develop general guidelines as to the level of strain for which the second-order approximate constitutive relations can be used to model certain types of elastic materials. The fact that second-order constitutive equations formulated in terms of the displacement gradient are not frame indifferent is strong reason to immediately reject them for any analysis involving finite deformations. However, the use of such constitutive equations may be attractive due to the ease with which they can be implemented. So we include constitutive equations which are second order in the displacement gradient in our presentation, and illustrate both types of problems for which such equations can be effectively used despite their limitations, and the large errors that result from their indiscriminant application in finite deformation problems.

Any discussion of the utility of a specific approximate constitutive equation for solving boundary value problems implicitly relies on the idea that the solution can be compared to a 'correct' solution that is independently obtained. Of course, given a real material, this would be done by comparing the predictions obtained with the approximate constitutive equation to the results of the corresponding experiments. This is the only approach that would give a definitive answer to whether or not a given constitutive equation is capable of making sufficiently accurate predictions for a real material over a specific range of deformations.

In this paper we do not attempt to obtain such definitive answers for specific materials. Rather, we are interested in formulating general guidelines that can aid in identifying problems and materials for which a second-order constitutive equation may be adequate. To that end, we will compare solutions of several boundary value problems obtained with fully nonlinear elastic constitutive equations to the solutions obtained with second-order approximations to those constitutive equations.

Two nonlinear compressible hyperelastic constitutive equations will be considered in this paper: the generalized Blatz–Ko material and the harmonic material. Compressible materials were selected for consideration because many of the real materials for which second-order theories may be of interest are compressible. These specific compressible materials were chosen because they have very different mechanical characteristics and because they are commonly used in the literature. For each of these fully nonlinear constitutive equations, three distinct second-order approximations for the Cauchy stress will be constructed: one that is second order in the displacement gradient, one that is second order in the Biot strain, and one that is second order in the Green strain. These constitutive equations will be used to solve four boundary value problems. For each problem the displacement fields and the Cauchy stresses predicted with the second-order constitutive equations will be compared to those calculated with the corresponding fully nonlinear constitutive relation.

Section 2 contains a brief review of background material. In Section 3 the nonlinear constitutive equations for the Blatz–Ko and harmonic materials will be displayed and the corresponding second-order constitutive equations will be derived. The performance of these second-order constitutive equations will be investigated for four boundary value problems in Sections 4–7. The problems are simple tension of a bar, axial shear of a cylinder, circular shear of a cylinder, and bending of a cantilever beam. The results and their implications will be discussed in Section 8.

2. Background

This section contains a review of the background information needed for the remainder of the paper. Standard notation is used throughout (see, for example, Gurtin 1984; Malvern, 1969).

2.1. Kinematics

Let $\mathscr{B}_0$ represent a body in a fixed reference configuration in which it is unloaded and at rest. A typical particle occupies position $\mathbf{X}$ in this reference configuration. When the body deforms due to prescribed tractions or displacements, the position of the particle originally at $\mathbf{X}$ is given by $\mathbf{x}=\mathbf{f}(\mathbf{X})$. The deformation $\mathbf{f}$ is a smooth one-to-one mapping. The displacement $\mathbf{u}(\mathbf{X})$ of the particle is defined by $\mathbf{u}(\mathbf{X})=\mathbf{f}(\mathbf{X})-\mathbf{X}$.

The deformation gradient

$$\mathbf{F}(\mathbf{X}) = \nabla\mathbf{f}(\mathbf{X}) \tag{1}$$

is assumed to meet det $\mathbf{F}(\mathbf{X}) > 0$. Unless required for clarity, dependence of a field on $\mathbf{X}$ will be left implicit in the remainder of this paper.

By the polar decomposition theorem, $\mathbf{F}$ can be written as

$$\mathbf{F} = \mathbf{R}\mathbf{U} \tag{2}$$

where the rotation $\mathbf{R}$ is proper orthogonal, and the right stretch tensor $\mathbf{U}$ is positive definite and symmetric. The eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ of $\mathbf{U}$ are called the principal stretches, and describe the ratio of the deformed length to the original length of a material filament in the principal directions.

Many constitutive equations are conveniently represented in terms of the left Cauchy–Green tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathrm{T}}$, which can be expressed as

$$\mathbf{B} = \mathbf{R}\mathbf{U}^2\mathbf{R}^{\mathrm{T}}. \tag{3}$$

The displacement gradient $\mathbf{H}=\nabla\mathbf{u}$ is related to the deformation gradient by

$$\mathbf{H} = \mathbf{F} - \mathbf{1}. \tag{4}$$

The symmetric part of $\mathbf{H}$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}) \tag{5}$$

is termed the elongation tensor (Truesdell and Toupin, 1960). $\mathbf{E}$ is most useful when the displacement gradient is small; then it is known as the infinitesimal strain tensor. But when the displacement gradient is not small, the elongation tensor is not a measure of strain as it includes a contribution from the rotation (see, e.g., Truesdell and Toupin, 1960).

In this paper we will employ two strain measures. The Biot strain, $\mathbf{E}_1$, is defined in terms of the right stretch tensor as

$$\mathbf{E}_1 = \mathbf{U} - \mathbf{1}. \tag{6}$$

The Biot strain has a direct physical interpretation in that its eigenvalues are the principal extensions $\{\delta_1, \delta_2, \delta_3\}$, defined through the principal stretches as

$$\delta_i = \lambda_i - 1. \tag{7}$$

We will also use the Green strain, $\mathbf{E}_2$, defined by

$$\mathbf{E}_2 = \tfrac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}). \tag{8}$$

The Green strain can be written in terms of the Biot strain as

$$\mathbf{E}_2 = \mathbf{E}_1 + \tfrac{1}{2}(\mathbf{E}_1)^2. \tag{9}$$

As the magnitude of the Biot strain tensor approaches zero, the Green strain approaches the Biot strain.

The elongation tensor $\mathbf{E}$ can also be related to the Biot strain, but, in contrast to the relation between $\mathbf{E}_2$ and $\mathbf{E}_1$, this relation depends on the rotation $\mathbf{R}$. To derive this relation, we introduce the tensor $\mathbf{P}$ defined by

$$\mathbf{P} = \mathbf{R} - \mathbf{1}. \tag{10}$$

While mathematically convenient, this tensor has no clear physical interpretation other than through its relation to $\mathbf{R}$. Note that if the rotation tensor $\mathbf{R}$ is close to the identity, then $|\mathbf{P}|$ will be small. The elongation tensor can now be written as (see Hoger, 1993),

$$\mathbf{E} = \mathbf{E}_1 + \tfrac{1}{2}(\mathbf{P} + \mathbf{P}^{\mathrm{T}}) + \tfrac{1}{2}(\mathbf{P}\mathbf{E}_1 + \mathbf{E}_1\mathbf{P}^{\mathrm{T}}). \tag{11}$$

This relation clearly displays the contribution of the rotation tensor to the elongation tensor.

The principal moments of a tensor $\mathbf{A}$ are defined through

$$I_1(\mathbf{A}) = \mathbf{1} \cdot \mathbf{A}$$

$$I_2(\mathbf{A}) = \mathbf{1} \cdot \mathbf{A}^2$$

$$I_3(\mathbf{A}) = \mathbf{1} \cdot \mathbf{A}^3 \tag{12}$$

where $\cdot$ indicates the inner product. The set of principal moments will be denoted by

$$\mathfrak{I}_{\mathbf{A}} = \{I_1(\mathbf{A}), I_2(\mathbf{A}), I_3(\mathbf{A})\}; \tag{13}$$

note that for $\mathbf{A} = \mathbf{1}$,

$$\mathfrak{I}_{\mathbf{1}} = \{3, 3, 3\}. \tag{14}$$

The principal invariants are defined in terms of the principal moments by

$$I_{\mathrm{A}} = I_1(\mathbf{A})$$

$$II_{\mathrm{A}} = \tfrac{1}{2}\{I_1(\mathbf{A})^2 - I_2(\mathbf{A})\}$$

$$III_{\mathrm{A}} = \tfrac{1}{6}\{I_1(\mathbf{A})^3 - 3I_1(\mathbf{A})I_2(\mathbf{A}) + 2I_3(\mathbf{A})\}. \tag{15}$$

2.2. Stress and strain energy density function

We assume the body responds elastically to deformations out of the reference configuration, so the constitutive equation for the Cauchy stress $\mathbf{T}$ can be written as a function of the deformation gradient in terms of a response function $\hat{\mathbf{T}}$ as

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}). \tag{16}$$

The first Piola–Kirchhoff stress tensor $\mathbf{S}$ is defined in terms of $\mathbf{T}$ through

$$\mathbf{S} = (\mathbf{det}\ \ \mathbf{F})\mathbf{T}\mathbf{F}^{-\mathrm{T}} =: \hat{\mathbf{S}}(\mathbf{F}). \tag{17}$$

A material is said to be hyperelastic if there exists a scalar valued function $\bar{\sigma}$, called the strain energy density function, such that

$$\hat{\mathbf{S}}(\mathbf{F}) = \frac{\partial \bar{\sigma}(\mathbf{F})}{\partial \mathbf{F}}. \tag{18}$$

2.3. Second-order constitutive theories

Various second-order constitutive theories have been derived for isotropic hyperelastic materials. These second-order constitutive theories are obtained by expanding the fully nonlinear constitutive relation for the Cauchy stress[2] in terms of a specified measure of the deformation, and neglecting terms which are higher than second order in this measure of deformation. Thus, in a general sense, these second-order constitutive theories will be good approximations to the fully nonlinear constitutive equation as long as the measure of the deformation is sufficiently small. To make this more precise, we need to establish some notation.

Let $\mathbf{M}$ be a function that maps tensors into tensors. Then we will say that $\mathbf{M(X)}$ approaches zero faster than $\mathbf{X}$ and write $\mathbf{M(X)} = o(\mathbf{X})$ as $\mathbf{X} \to \mathbf{0}$ if $\lim_{\mathbf{X}\to 0} |\mathbf{M(X)}|/|\mathbf{X}| = 0$, where the magnitude of a tensor $\mathbf{A}$ is defined through the inner product as $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$.

The most widely recognized second-order constitutive theory for isotropic hyperelastic materials is one that uses the displacement gradient as the measure of deformation. A number of authors have presented this theory in equivalent forms (Truesdell and Noll, 1965; Lindsay, 1992; Haughton and Lindsay, 1993, 1994; Rivlin, 1952; Murnaghan, 1937; Toupin and Bernstein, 1961; Hoger, 1998). For our purposes, it is convenient to use the form (Hoger, 1998)[3]

$$\begin{aligned}\mathbf{T_H} = {} & \eta_1(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + \eta_2\mathbf{E} + (\eta_5 - \eta_1)(\mathbf{1}\cdot\mathbf{E})^2\mathbf{1} + (\eta_4 - \tfrac{1}{2}\eta_1)(\mathbf{1}\cdot\mathbf{E}^2)\mathbf{1} + (2\eta_4 - \eta_2 + \eta_1)(\mathbf{1}\cdot\mathbf{E})\mathbf{E} \\ & + (\eta_3 - \tfrac{3}{2}\eta_2)\mathbf{E}^2 + \tfrac{1}{2}\eta_1(\mathbf{1}\cdot\mathbf{H}^{\mathrm{T}}\mathbf{H})\mathbf{1} + \eta_2(\mathbf{HE} + \mathbf{EH}^{\mathrm{T}} + \tfrac{1}{2}\mathbf{H}^{\mathrm{T}}\mathbf{H}) + o(\mathbf{H}^2)\end{aligned} \tag{19}$$

[2] Of course a constitutive equation can be written for other measures of stress, such as the first and second Piola–Kirchhoff stresses. Expansion of these stresses will result in different second-order constitutive equations, as was recognised and illustrated by Ogden (1984). In this paper, we use the expansion for the Cauchy stress because of its clear interpretation as the physical stress in the deformed body. Although we do not use the expansions of the other stress measures in the examples we present in Section 4–7, we did do the examples with expansions of both the first and second Piola–Kirchhoff stresses. We will comment on those results in the discussion in Section 8.

[3] It can be shown that the constants given in (19) are related to those given in Eq. (66.3) of Truesdell and Noll (1965) by $\eta_1 = \mu\alpha_1$, $\eta_2 = 2\mu\alpha_2$, $\eta_3 = \mu(\alpha_6 - \alpha_2)$, $\eta_4 = \frac{1}{2}\mu(\alpha_5 + 2\alpha_2 - \alpha_1)$, and $\eta_5 = \mu(2\alpha_1 - \alpha_2 + \alpha_3 - \alpha_5)$. To demonstrate the equivalency of the two forms, it is necessary to use $\alpha_4 + \alpha_5 = 2\alpha_1 + 2\alpha_2$, which is required by the assumption of hyperelasticity.

where

$$\eta_1 = \frac{\partial^2\bar{\sigma}}{\partial I_1\,\partial I_1} + 4\frac{\partial^2\bar{\sigma}}{\partial I_2\,\partial I_2} + 9\frac{\partial^2\bar{\sigma}}{\partial I_3\,\partial I_3} + 4\frac{\partial^2\bar{\sigma}}{\partial I_1\,\partial I_2} + 6\frac{\partial^2\bar{\sigma}}{\partial I_1\,\partial I_3} + 12\frac{\partial^2\bar{\sigma}}{\partial I_2\,\partial I_3}$$

$$\eta_2 = 2\frac{\partial\bar{\sigma}}{\partial I_2} + 6\frac{\partial\bar{\sigma}}{\partial I_3}$$

$$\eta_3 = 3\frac{\partial\bar{\sigma}}{\partial I_3}$$

$$\eta_4 = 2\frac{\partial^2\bar{\sigma}}{\partial I_2\,\partial I_2} + 9\frac{\partial^2\bar{\sigma}}{\partial I_3\,\partial I_3} + \frac{\partial^2\bar{\sigma}}{\partial I_1\,\partial I_2} + 3\frac{\partial^2\bar{\sigma}}{\partial I_1\,\partial I_3} + 9\frac{\partial^2\bar{\sigma}}{\partial I_2\,\partial I_3}$$

$$\begin{aligned}\eta_5 = \frac{1}{2}\Bigg\{ & \frac{\partial^3\bar{\sigma}}{\partial I_1\,\partial I_1\,\partial I_1} + 6\frac{\partial^3\bar{\sigma}}{\partial I_1\,\partial I_1\,\partial I_2} + 9\frac{\partial^3\bar{\sigma}}{\partial I_1\,\partial I_1\,\partial I_3} + 12\frac{\partial^3\bar{\sigma}}{\partial I_1\,\partial I_2\,\partial I_2} + 36\frac{\partial^3\bar{\sigma}}{\partial I_1\,\partial I_2\,\partial I_3} \\ & + 27\frac{\partial^3\bar{\sigma}}{\partial I_1\,\partial I_3\,\partial I_3} + 8\frac{\partial^3\bar{\sigma}}{\partial I_2\,\partial I_2\,\partial I_2} + 36\frac{\partial^3\bar{\sigma}}{\partial I_2\,\partial I_2\,\partial I_3} + 54\frac{\partial^3\bar{\sigma}}{\partial I_2\,\partial I_3\,\partial I_3} + 27\frac{\partial^3\bar{\sigma}}{\partial I_3\,\partial I_3\,\partial I_3}\Bigg\}\end{aligned} \tag{20}$$

and where all terms are evaluated on the set of principal moments at $\mathbf{H}=\mathbf{0}$, that is, on $\mathfrak{J}_{\mathbf{1}}$. The subscript $\mathbf{H}$ on the stress indicates that this constitutive equation is obtained by expanding the finite constitutive equation in the deformation gradient. For convenience, this equation will be referred to as the $\mathbf{T_H}$ *theory*.

Recently Hoger (1998) derived a second-order constitutive theory for an isotropic hyperelastic material in which the Biot strain is used as the measure of deformation. The resulting second-order constitutive relation for Cauchy stress is

$$\begin{aligned}\mathbf{T}_{\mathbf{E}_1} = \mathbf{R}\{ & \eta_1(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + \eta_2\mathbf{E}_1 + (\eta_2+\eta_3)\mathbf{E}_1^2 + \eta_4(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{1} + (\eta_1-\eta_2+2\eta_4)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1 \\ & + (\eta_5-\eta_1)(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}} + o(\mathbf{E}_1^2)\end{aligned} \tag{21}$$

where the coefficients η_i are given in (20). This equation will be referred to as the $\mathbf{T}_{\mathbf{E}_1}$ *theory*. It can be shown that for deformations with $\mathbf{R}=\mathbf{1}$ (so $\mathbf{P}=\mathbf{0}$), the $\mathbf{T}_{\mathbf{E}_1}$ theory becomes identical to the $\mathbf{T_H}$ theory. It can also be shown that if the displacement gradient is small (which implies that both the Biot strain and the tensor $\mathbf{P}$ are small), the two constitutive theories will be within $o(\mathbf{H}^2)$. Thus, as expected, the two constitutive theories will give similar results if not only is $\mathbf{E}_1$ small, as is assumed in the development of the $\mathbf{T}_{\mathbf{E}_1}$ theory, but $\mathbf{P}$ is restricted as well.

Murnaghan (1951) developed a second-order constitutive equation for isotropic hyperelastic materials in terms of the Green strain. His expression is equivalent to

$$\begin{aligned}\mathbf{T}_{\mathbf{E}_2} = \mathbf{R}\{ & \eta_1(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{1} + \eta_2\mathbf{E}_2 + (\tfrac{1}{2}\eta_2+\eta_3)\mathbf{E}_2^2 + (\eta_4-\tfrac{1}{2}\eta_1)(\mathbf{1}\cdot\mathbf{E}_2^2)\mathbf{1} + (\eta_1-\eta_2+2\eta_4)(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{E}_2 \\ & + (\eta_5-\eta_1)(\mathbf{1}\cdot\mathbf{E}_2)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}} + o(\mathbf{E}_2^2)\end{aligned} \tag{22}$$

where, again, the coefficients η_i are defined in (20). This constitutive equation will be referred to as the $\mathbf{T}_{\mathbf{E}_2}$ *theory*. The relation between the $\mathbf{T}_{\mathbf{E}_1}$ theory and the $\mathbf{T}_{\mathbf{E}_2}$ theory was derived by Hoger (1998), who showed that if $\mathbf{E}_1$ is sufficiently small, the $\mathbf{T}_{\mathbf{E}_1}$ theory and the $\mathbf{T}_{\mathbf{E}_2}$ theory are equal to within $o(\mathbf{E}_1^2)$. Of course, this result does not require that the rotations be restricted in any way.

3. Constitutive equations

The second-order constitutive theory $\mathbf{T}_\mathbf{H}$ presented in the last section is valid only in the limit as $\mathbf{H}$ approaches zero. By neglecting the terms which are $o(\mathbf{H}^2)$, we can define the corresponding *St Venant–Kirchhoff-type material* that is second order in $\mathbf{H}$. This St Venant–Kirchhoff material model, which can be used for deformations with arbitrarily large displacement gradients, will be referred to as the $\mathbf{T}_\mathbf{H}$ *constitutive equation* to distinguish it from the $\mathbf{T}_\mathbf{H}$ *constitutive theory*, given by (19), which can be applied only when $\mathbf{H}$ is sufficiently small. Second-order St Venant–Kirchhoff–type constitutive equations corresponding to the $\mathbf{T}_{\mathbf{E}_1}$ and $\mathbf{T}_{\mathbf{E}_2}$ theories of Section 2.3 can be similarly constructed from the corresponding constitutive theories. In the remainder of the paper these St Venant–Kirchhoff–type constitutive equations will be termed simply the second-order constitutive equations.

The purpose of this paper is to investigate the circumstances under which these second-order constitutive equations can be used in lieu of a fully nonlinear constitutive relation for the solution of boundary value problems. To accomplish this, we will solve a variety of boundary value problems with the three second-order constitutive equations and compare the solutions they produce to the solutions obtained by solving the same problems with the corresponding fully nonlinear constitutive equation.

We will consider two nonlinear elastic materials: generalized Blatz–Ko and harmonic. We chose these two materials for study because they have quite different material properties and because they are commonly used in the literature. In this section, we recall the fully nonlinear constitutive equations for these two materials. Using these equations, we derive the elastic constants, η_i, of the second-order constitutive equations, and for each of the materials we display the constitutive equations which are second order in $\mathbf{H}$, $\mathbf{E}_1$, and $\mathbf{E}_2$.

3.1. Generalized Blatz–Ko material

The generalized Blatz–Ko material is a generalization of two constitutive relations that were obtained experimentally by Blatz and Ko (1962). This constitutive equation can also be derived theoretically by making a few simple assumptions about the nature of the material (see Beatty and Stalnaker, 1986; Beatty, 1987).

The strain-energy density function for the isotropic, generalized Blatz–Ko material is

$$\bar{\sigma}(I_\mathrm{B}, II_\mathrm{B}, III_\mathrm{B}) = \frac{\mu_0 f}{2}\left[(I_\mathrm{B}-3) - \frac{2}{q}(III_\mathrm{B}^{q/2}-1)\right] + \frac{\mu_0(1-f)}{2}\left[\left(\frac{II_\mathrm{B}}{III_\mathrm{B}}-3\right) - \frac{2}{q}(III_\mathrm{B}^{-q/2}-1)\right] \tag{23}$$

where μ_0 is the infinitesimal shear modulus,

$$q = -\frac{2\nu_0}{1-2\nu_0} \tag{24}$$

with ν_0 the infinitesimal Poisson ratio, and f is a material parameter between 0 and 1. Thus, by (17) and (18), the Cauchy stress is

$$\mathbf{T} = \mu_0[-f III_\mathrm{B}^{(q-1)/2} + (1-f) III_\mathrm{B}^{-(q-1)/2}]\mathbf{1} + \frac{\mu_0 f}{III_\mathrm{B}^{1/2}}\mathbf{B} - \frac{\mu_0(1-f)}{III_\mathrm{B}^{1/2}}\mathbf{B}^{-1}. \tag{25}$$

This equation will be referred to as the *fully nonlinear generalized Blatz–Ko constitutive equation.*

All of the results that are presented in this paper with the generalized Blatz–Ko material are computed with

$$\nu_0 = 0.25 \tag{26}$$

which is a value experimentally determined by Blatz and Ko for a polyurethane foam rubber.

By substituting (23) into (20), we obtain the following coefficients for the generalized Blatz–Ko material:

$$\eta_1 = -\mu_0 q$$

$$\eta_2 = 2\mu_0$$

$$\eta_3 = \mu_0(4f - 5)$$

$$\eta_4 = \tfrac{1}{2}\mu_0 q$$

$$\eta_5 = \tfrac{1}{2}\mu_0 q^2(1 - 2f\,). \tag{27}$$

Using these coefficients, we can obtain a second-order constitutive equation for the generalized Blatz–Ko material from (19); the result is

$$\mathbf{T_H} = -\mu_0 q(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + 2\mu_0\mathbf{E} + \mu_0\left(q + \frac{1}{2}q^2(1-2f\,)\right)(\mathbf{1}\cdot\mathbf{E})^2\mathbf{1} + \mu_0 q(\mathbf{1}\cdot\mathbf{E}^2)\mathbf{1} - 2\mu_0(\mathbf{1}\cdot\mathbf{E})\mathbf{E}$$
$$+ 4\mu_0(f-2)\mathbf{E}^2 - \frac{\mu_0 q}{2}(\mathbf{1}\cdot\mathbf{H}^{\mathrm{T}}\mathbf{H})\mathbf{1} + 2\mu_0\left(\mathbf{HE} + \mathbf{EH}^{\mathrm{T}} + \frac{1}{2}\mathbf{H}^{\mathrm{T}}\mathbf{H}\right). \tag{28}$$

This equation will be referred to as the $\mathbf{T_H}$ *constitutive equation for the generalized Blatz–Ko material.*

Similarly, we obtain the $\mathbf{T}_{\mathbf{E}_1}$ *constitutive equation for the generalized Blatz–Ko material* by incorporating (27) into (21) to get

$$\mathbf{T}_{\mathbf{E}_1} = \mathbf{R}\Bigg\{ -\mu_0 q(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + 2\mu_0\mathbf{E}_1 + \mu_0(4f-3)\mathbf{E}_1^2 + \frac{\mu_0 q}{2}(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{1} + 2\mu_0(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1$$
$$+ \mu_0\left(q + \frac{1}{2}q^2(1-2f\,)\right)(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}\Bigg\}\mathbf{R}^{\mathrm{T}} \tag{29}$$

and we find the $\mathbf{T}_{\mathbf{E}_2}$ *constitutive equation for the generalized Blatz–Ko material* by using (27) with (22) with the result

$$\mathbf{T}_{\mathbf{E}_2} = \mathbf{R}\{-\mu_0 q(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{1} + 2\mu_0\mathbf{E}_2 + 4\mu_0(f-1)\mathbf{E}_2^2 + \mu_0 q(\mathbf{1}\cdot\mathbf{E}_2^2)\mathbf{1} - 2\mu_0(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{E}_2$$
$$+ \mu_0(q + \tfrac{1}{2}q^2(1-2f\,))(\mathbf{1}\cdot\mathbf{E}_2)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}}. \tag{30}$$

3.2. Harmonic material

The constitutive relation for harmonic materials was originally developed by John (1960, 1966), and was motivated primarily by mathematical convenience rather than experimental evidence. We consider this material because it has very different properties than those of the generalized Blatz–Ko material and because it has previously been used by a number of authors (see, for example, Ogden and Isherwood, 1978; Abeyaratne and Horgan, 1984; Jafari et al., 1984).

The isotropic strain energy density function proposed by John (1960, 1966) has the form

$$\bar{\sigma}(I_{\mathrm{U}}, II_{\mathrm{U}}, III_{\mathrm{U}}) = 2\mu_0[F(I_{\mathrm{U}}) - III_{\mathrm{U}} + 1] \tag{31}$$

where μ_0 is the infinitesimal shear modulus and F is a scalar function that must satisfy certain inequalities (see Knowles and Sternberg, 1975a), but is otherwise arbitrary. Following Haughton and Lindsay (1993), we will take

$$F(I_{\mathrm{U}}) = I_{\mathrm{U}} - 3 + \alpha(I_{\mathrm{U}} - 3)^2 \tag{32}$$

where α is a material parameter given in terms of the Poisson ratio ν_0 by

$$\alpha = \frac{1 - \nu_0}{2(1 - 2\nu_0)}. \tag{33}$$

All results that will be presented in this paper for the harmonic material were obtained with

$$\nu_0 = 0.40625 \tag{34}$$

following Haughton and Lindsay (1993).

By (17) and (18) the Cauchy stress for this material is

$$\mathbf{T} = 2\mu_0\mathbf{R}\left\{-\mathbf{1} + \frac{1 + 2\alpha(I_{\mathrm{U}} - 3)}{III_{\mathrm{U}}}\mathbf{U}\right\}\mathbf{R}^{\mathrm{T}}. \tag{35}$$

This equation will be referred to as the *fully nonlinear constitutive equation for the harmonic material*.

By substituting (31) and (32) into (20), we obtain:

$$\eta_1 = 2\mu_0(2\alpha - 1)$$

$$\eta_2 = 2\mu_0$$

$$\eta_3 = -2\mu_0$$

$$\eta_4 = \mu_0$$

$$\eta_5 = -\mu_0. \tag{36}$$

By using these coefficients with (19), we obtain the $\mathbf{T_H}$ *constitutive equation for the harmonic material*:

$$\begin{aligned}\mathbf{T_H} = {} & 2\mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{E})\mathbf{1} + 2\mu_0\mathbf{E} + \mu_0(1 - 4\alpha)(\mathbf{1}\cdot\mathbf{E})^2\mathbf{1} + 2\mu_0(1 - \alpha)(\mathbf{1}\cdot\mathbf{E}^2)\mathbf{1} \\ & + 2\mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{E})\mathbf{E} - 5\mu_0\mathbf{E}^2 + \mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{H}^{\mathrm{T}}\mathbf{H})\mathbf{1} + 2\mu_0(\mathbf{HE} + \mathbf{EH}^{\mathrm{T}} + \tfrac{1}{2}\mathbf{H}^{\mathrm{T}}\mathbf{H}).\end{aligned} \tag{37}$$

Eq. (36) together with (21) gives the $\mathbf{T}_{\mathbf{E}_1}$ *constitutive equation for the harmonic material*:

$$\mathbf{T}_{\mathbf{E}_1} = \mathbf{R}\{2\mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{1} + 2\mu_0\mathbf{E}_1 + \mu_0(\mathbf{1}\cdot\mathbf{E}_1^2)\mathbf{1} + 2\mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{E}_1)\mathbf{E}_1 + \mu_0(1 - 4\alpha)(\mathbf{1}\cdot\mathbf{E}_1)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}}. \tag{38}$$

Substitution of (36) into (22) provides the $\mathbf{T}_{\mathbf{E}_2}$ *constitutive equation for the harmonic material*:

$$\mathbf{T}_{\mathbf{E}_2} = \mathbf{R}\{2\mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{1} + 2\mu_0\mathbf{E}_2 - \mu_0\mathbf{E}_2^2 + 2\mu_0(1-\alpha)(\mathbf{1}\cdot\mathbf{E}_2^2)\mathbf{1} + 2\mu_0(2\alpha - 1)(\mathbf{1}\cdot\mathbf{E}_2)\mathbf{E}_2 + \mu_0(1 - 4\alpha)(\mathbf{1}\cdot\mathbf{E}_2)^2\mathbf{1}\}\mathbf{R}^{\mathrm{T}}. \tag{39}$$

4. Simple tension of a bar

As our first boundary value problem, we consider simple tension of a bar, a problem which can be solved in closed form. Let the body be an isotropic homogeneous rectangular bar with constant cross section and length L. Without loss of generality, we take the x-axis of a Cartesian coordinate system to coincide with the axis of the beam and fix the origin of this system on one of the faces perpendicular to this axis. This face is then held fixed against motion in the x-direction and is otherwise unloaded. The face at the opposite end is subjected to a uniform displacement in the x-direction and is otherwise unloaded. The remaining faces are all traction free.

With the displacement field denoted by $\mathbf{u}(X, Y, Z)$, where (X, Y, Z) is the initial location of a particle, the displacement boundary condition on the fixed face is

$$\mathbf{u}(0, Y, Z)\cdot\mathbf{e}_x = 0 \tag{40}$$

where $\mathbf{e}_x$ is a unit vector in the x-direction. The displacement boundary condition on the displaced face is

$$\mathbf{u}(L, Y, Z)\cdot\mathbf{e}_x = \bar{u} \tag{41}$$

with L the length of the bar and $\bar{u}$ the prescribed displacement.

Because the material is isotropic, we assume a displacement of the form

$$\begin{aligned} u_x &= (\lambda_1 - 1)X \\ u_y &= (\lambda_2 - 1)Y \\ u_z &= (\lambda_2 - 1)Z. \end{aligned} \tag{42}$$

With these displacements,

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \tag{43}$$

$\mathbf{U} = \mathbf{F}$, and $\mathbf{R} = \mathbf{1}$,

Since the deformation is uniform, the stresses are constant throughout the bar, and the equation of equilibrium is automatically satisfied.

Boundary condition (41) requires that the principal stretch satisfy

$$\lambda_1 = \frac{\bar{u}}{L} + 1. \tag{44}$$

4.1. Generalized Blatz–Ko material

The results for various values of the material parameter f in the generalized Blatz–Ko material are similar, so only the results for $f = 1$ will be presented.

4.1.1. The fully nonlinear constitutive equation

The solution will first be obtained for the fully nonlinear generalized Blatz–Ko material. Substituting (43) into (25), we find the components of the Cauchy stress to be

$$T_{xx} = \mu_0[\lambda_1^2 - (\lambda_1\lambda_2^2)^q]\frac{1}{\lambda_1\lambda_2^2}$$

$$T_{yy} = T_{zz} = \mu_0[\lambda_2^2 - (\lambda_1\lambda_2^2)^q]\frac{1}{\lambda_1\lambda_2^2} \tag{45}$$

with all of the shear stresses being zero. As noted previously, the equilibrium equations are trivially satisfied. The zero traction boundary conditions on the lateral surface require that $T_{yy}=0$ and $T_{zz}=0$; therefore, since the denominator in $(45)_2$ cannot be 0,

$$\lambda_2 = \lambda_1^{q/[2(1-q)]} \tag{46}$$

where λ_1 is given by (44) to satisfy the boundary conditions. With (46) and $(45)_1$ yields

$$T_{xx} = \mu_0(\lambda_1^{(2q-1)/(q-1)} - \lambda_1^{-1}). \tag{47}$$

4.1.2. The $\boldsymbol{T_H}$ and $\boldsymbol{T_{E_l}}$ constitutive equations

Since $\mathbf{R} \equiv \mathbf{1}$ for this problem, constitutive equations $\mathbf{T_H}$ and $\mathbf{T_{E_l}}$ yield identical results.

With $\nu_0 = 0.25$ the $\mathbf{T_H}$ and the $\mathbf{T_{E_l}}$ constitutive equations for the generalized Blatz–Ko material, (28) and (29), give the following non-zero Cauchy stress components:

$$T_{xx} = \mu_0(-25 + 19\lambda_1 - 3\lambda_1^2 + 26\lambda_2 - 10\lambda_1\lambda_2 - 7\lambda_2^2)$$

$$T_{yy} = T_{zz} = \mu_0(-25 + 13\lambda_1 - 2\lambda_1^2 + 32\lambda_2 - 8\lambda_1\lambda_2 - 10\lambda_2^2). \tag{48}$$

Again, the equilibrium equations are trivially satisfied, and T_{yy} and T_{zz} must vanish by the zero traction boundary condition on the lateral surface. This condition can only be satisfied if

$$\lambda_2 = \frac{16 - 4\lambda_1 - \sqrt{6 + 2\lambda_1 - 4\lambda_1^2}}{10} \quad \text{or } \lambda_2 = \frac{16 - 4\lambda_1 + \sqrt{6 + 2\lambda_1 - 4\lambda_1^2}}{10}. \tag{49}$$

With the fully nonlinear generalized Blatz–Ko constitutive relation, there is a unique relation between λ_2 and λ_1. But here, for the $\mathbf{T_H}$ and the $\mathbf{T_{E_l}}$ constitutive equation, there are two possible relations. It is natural to expect that $\lambda_2 \to 1$ as $\lambda_1 \to 1$ (that is, as $\bar{u} \to 0$) for the solution to be physically meaningful. This condition is satisfied by the solution given by $(49)_1$; it is not the case for the solution given by $(49)_2$. In fact one can see from $(49)_2$, that $\lambda_2 \to 1.4$ as $\lambda_1 \to 1$. So, we will take λ_2 to be given by $(49)_1$.

Substitution of $(49)_1$ into $(48)_1$ gives

$$T_{xx} = \mu_0 \left(\frac{-87 + 71\lambda_1 + 8\lambda_1^2 + (-18 + 22\lambda_1)\sqrt{6 + 2\lambda_1 - 4\lambda_1^2}}{50} \right) \tag{50}$$

where λ_1 is fixed by the boundary condition through (44).

4.1.3. The $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation

Substitution of deformation gradient (43) into the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation for the generalized Blatz–Ko material, (30), gives non-zero stress components

$$T_{xx} = \frac{\mu_0}{8}(-65 + 50\lambda_1^2 - 9\lambda_1^4 + 60\lambda_2^2 - 20\lambda_1^2\lambda_2^2 - 16\lambda_2^4)$$

$$T_{yy} = T_{zz} = \frac{\mu_0}{8}(-65 + 30\lambda_1^2 - 5\lambda_1^4 + 80\lambda_2^2 - 16\lambda_1^2\lambda_2^2 - 24\lambda_2^4). \tag{51}$$

The equilibrium equations are trivially satisfied. The zero traction boundary condition on the lateral surface implies that T_{yy} and T_{zz} vanish, so $(51)_2$ produces four solutions for the relation between λ_2 and λ_1. Of these, two solutions give negative values for λ_2 which is not physically meaningful, and one gives large values for λ_2 even as λ_1 approaches 1, which is also not physically realistic. The remaining solution is

$$\lambda_2 = \frac{\sqrt{10 - 2\lambda_1^2 - \sqrt{\frac{5}{2} + 5\lambda_1^2 - \frac{7}{2}\lambda_1^4}}}{\sqrt{6}}. \tag{52}$$

By substituting this result for λ_2 into the formula for T_{xx} in (51) we obtain

$$T_{xx} = \mu_0 \left(\frac{-95 + 110\lambda_1^2 - 23\lambda_1^4 + (5 + 7\lambda_1^2)\sqrt{5 + 10\lambda_1^2 - 7\lambda_1^4}}{72} \right) \tag{53}$$

where, again, λ_1 is given in terms of the prescribed displacement by (44).

4.1.4. Comparison of results

The solutions for this boundary value problem are displayed in Fig. 1. The values of the transverse displacement, λ_2−1, obtained with the finite and the second-order generalized Blatz–Ko constitutive equations are displayed in the upper part of that figure, and the values of the normalized stress T_{xx}/μ_0 are displayed in the lower part. Recall that $\mathbf{R} = \mathbf{1}$ for this problem so $\mathbf{T}_\mathbf{H}$ and $\mathbf{T}_{\mathbf{E}_1}$ give the same solution. Of course, for sufficiently small values of $\bar{u}/L$, all of the results coalesce.

Both second-order constitutive equations fail to give real solutions for λ_2 when $\bar{u}$, and, therefore, λ_1, is larger than a certain critical value; for the $\mathbf{T}_\mathbf{H}$ and $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations this critical value is $\lambda_1 = 3/2$ and for the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation this critical value is $\lambda_1 = (1/\sqrt{7})\sqrt{5 + 2\sqrt{15}} \approx 1.35$. More precisely, the lateral traction boundary condition, $T_{yy} = 0$, yields complex values for λ_2 for values of λ_1 larger than these critical values[4]. As can be seen from Fig. 1, both the stress and the displacement predicted by the second-order constitutive equations radically diverge from those provided by the fully nonlinear constitutive equation well before these critical values are reached.

[4] The other solutions to the equation $T_{yy} = 0$ also become complex for values of λ_1 above this critical value.

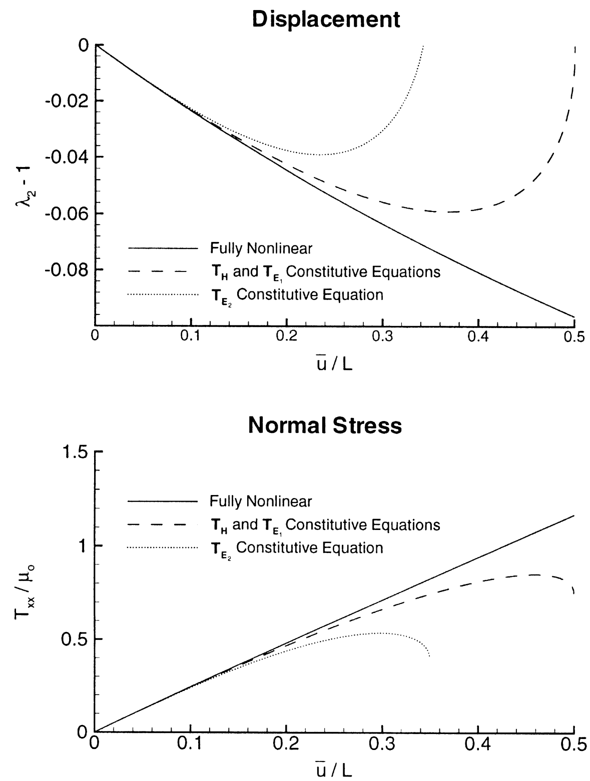

Fig. 1. Displacement and normal stress for a bar of generalized Blatz–Ko material with $f = 1$ in simple tension.

In order to compare the utility of the second-order constitutive equations, we will determine the percent differences in the solutions as follows. For each component of displacement and stress, we will calculate the difference between the value obtained with the second-order constitutive equation and the value obtained with the fully nonlinear equation. This difference will be divided by the value of the component obtained with the fully nonlinear constitutive equation, yielding the percent difference. These percent differences must, of course, be compared at the same level of the loading parameter, in this case, the same displacement $\bar{u}$. This loading parameter, however, is specific to the problem being solved and depends on the geometry of the body and the nature of the loading. It does not, therefore, lend itself directly to any understanding of the nature of the constitutive equations. On the other hand, the level of strain in the problem gives some idea of the response of the material itself. Therefore, we choose to display the percent differences in the displacement and stress components in terms of the magnitude of the Biot strain calculated from the fully nonlinear constitutive equation in all cases[5].

The percent difference in the displacement and the stress for simple tension of a bar are shown in Fig. 2. The $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations give substantially smaller percent differences than the $\mathbf{T_{E_2}}$ constitutive equation at all levels of strain. For example, for a Biot strain magnitude of 15%, the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations give a percent difference in the displacement of 2.2% while the $\mathbf{T_{E_2}}$

[5] For all the problems that are presented in this paper, the difference between the strain measures calculated using the results from the fully nonlinear constitutive equation and calculated from the results obtained from the second-order constitutive equation is less than 10% even at the largest level of strain.

constitutive equation gives a percent difference of 6.8%. The information in this figure can also be viewed in an alternative manner: given a particular value for the acceptable percent difference, the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations give an acceptable solution to a higher level of strain than does the $\mathbf{T_{E_2}}$ constitutive equation. If, for example, it is necessary to obtain solutions with less than 5% difference in both the displacement and the stress, the solution obtained with the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations is acceptable to a strain of approximately 21%, whereas the $\mathbf{T_{E_2}}$ constitutive equation fails to meet this level of accuracy after a strain of 13% has been reached.

The range of strains for which a second-order constitutive equation is a good model for the material will depend, of course, on the amount of percent difference that can be tolerated in the solution. We will use 5% difference throughout this paper only for illustration. If only general trends are needed, larger percent differences may be acceptable. If very accurate predictions are needed, then the second-order constitutive equations would be useful over a much smaller range of strains. Note that the curves in Fig. 2 do not cross, so the solution provided by the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations is more accurate than that of the $\mathbf{T_{E_2}}$ constitutive equation at any strain magnitude.

4.2. Harmonic material

The procedure for analyzing the extension of a homogeneous bar composed of harmonic material

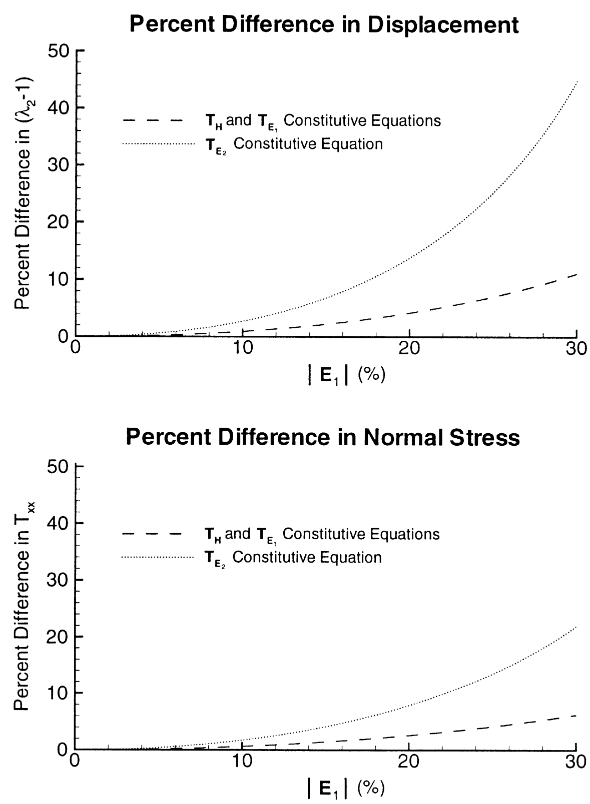

Fig. 2. Percent difference in displacement and normal stress for a bar of generalized Blatz–Ko material with $f = 1$ in simple tension.

parallels the analysis presented above for a bar made of the generalized Blatz–Ko material, so details will be omitted.

4.2.1. The fully nonlinear constitutive equation

By substituting (43) into (35) and using the zero traction condition on the lateral surfaces, we find that for a homogeneous bar of harmonic material

$$\lambda_2 = \frac{1 - 6\alpha + 2\alpha\lambda_1}{\lambda_1 - 4\alpha} \quad \text{and} \quad T_{xx} = -2\mu_0 + \frac{2\mu_0\lambda_1(\lambda_1 - 4\alpha)}{1 - 6\alpha + 2\alpha\lambda_1} \tag{54}$$

where the boundary condition (41) requires that λ_1 meet (44).

4.2.2. The T_H and T_{E_1} constitutive equations

As previously noted, since $\mathbf{R} = \mathbf{1}$ for this deformation, the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations give identical results. When the deformation gradient, (43), is substituted into either (37) or (38) and the zero traction condition on the lateral surfaces is applied, the only physically realistic choice of λ_2 is found to be

$$\lambda_2 = \frac{1}{8\alpha - 2}[-4 + 18\alpha + \lambda_1 - 6\alpha\lambda_1 - (-4 + 8\alpha + 36\alpha^2 + 8\lambda_1 - 28\alpha\lambda_1 - 24\alpha^2\lambda_1 - 3\lambda_1^2 + 12\alpha\lambda_1^2 + 4\alpha^2\lambda_1^2)^{1/2}] \tag{55}$$

and the corresponding expression for T_{xx} is

$$T_{xx} = 2\mu_0(5 - 18\alpha + 6\alpha\lambda_1 - 8\lambda_2 + 24\alpha\lambda_2 - 4\alpha\lambda_1\lambda_2 + 3\lambda_2^2 - 8\alpha\lambda_2^2). \tag{56}$$

4.2.3. The T_{E_2} constitutive equation

Combining (39) and (43), and the zero traction condition on the lateral surfaces, we obtain

$$\lambda_2 = \frac{1}{\sqrt{3 - 12\alpha}}[6 - 26\alpha - \lambda_1^2 + 6\alpha\lambda_1^2 + (-12 + 42\alpha + 28\alpha^2 + 24\lambda_1^2 - 116\alpha\lambda_1^2 + 72\alpha^2\lambda_1^2 - 8\lambda_1^4 + 42\alpha\lambda_1^4 - 38\alpha^2\lambda_1^4)^{1/2}]^{1/2} \tag{57}$$

where non-physical choices of λ_2 have been neglected, and we also obtain

$$T_{xx} = \mu_0(4 - \tfrac{27}{2}\alpha + 5\alpha\lambda_1^2 - \tfrac{1}{2}\lambda_1^4 - 6\lambda_2^2 + 16\alpha\lambda_2^2 - 2\alpha\lambda_1^2\lambda_2^2 + 2\lambda_2^4 - 5\alpha\lambda_2^4). \tag{58}$$

4.2.4. Comparison of results

The percent differences in the predicted displacement and the predicted stress T_{xx} obtained with the second-order constitutive equations are shown in Fig. 3. For the harmonic material, the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations give more accurate solutions than does the $\mathbf{T_{E_2}}$ constitutive equation for both the displacement and the stress, paralleling the results obtained for the generalized Blatz–Ko material. However, for harmonic materials all percent differences are much smaller: at a strain of 15%, the percent difference in the displacement from the $\mathbf{T_H}$ and $\mathbf{T_{E_1}}$ constitutive equations is 0.2% while for the $\mathbf{T_{E_2}}$ constitutive equation, the percent difference is 1.3%. Further, the percent differences in the components of both the displacement and the stress remain smaller than 5% for strain in excess of 25%.

Recall from Section 4.1 that for simple tension of a bar made of generalized Blatz–Ko material, each second-order constitutive equation leads to governing equations which have no real solutions for strains above some critical value. Similarly, for simple tension of a bar composed of harmonic material, the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation leads to governing equations which have no real solutions for strains above $|\mathbf{E}_1| = 63\%$. But the $\mathbf{T}_\mathbf{H}$ and $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations lead to governing equations which have real solutions for arbitrarily large strains.

4.3. Summary: simple tension of a bar

The extremely simple problem of a homogeneous bar in tension illustrates several points:

- As expected, when the deformation involves no rotation, the $\mathbf{T}_\mathbf{H}$ and $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations become identical and, therefore, the solutions are the same.
- The magnitude of the percent difference in the solution obtained with the second-order constitutive equations depends on the nature of the material. The percent differences for a bar composed of a harmonic material are much smaller than those calculated for a bar made of a generalized Blatz–Ko material. The relative performance of the various second-order constitutive equations, however, remained similar for this problem.
- Solutions obtained with constitutive equations that are second order in different measures of strain can vary substantially. For the case of a bar in simple tension, the $\mathbf{T}_\mathbf{H}$ and $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations produce solutions with substantially smaller percent differences than does the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation. This observation holds true for both the generalized Blatz–Ko and the harmonic materials.

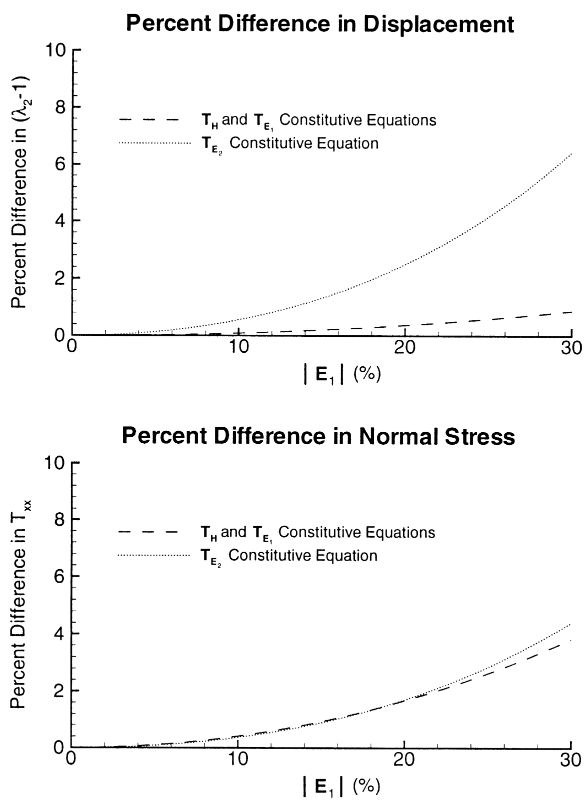

Fig. 3. Percent difference in displacement and normal stress for a bar of harmonic material in simple tension.

- If a second-order constitutive equation is used to solve a problem, there may be levels of strain above which the equations have no physically meaningful solutions. This may be the case even if there are physically meaningful solutions to the problem when it is solved using the fully nonlinear constitutive equation at the same levels of strain. This was seen for both the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation and the $\mathbf{T}_{\mathbf{H}}$ and $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations; however, the $\mathbf{T}_{\mathbf{H}}$ and $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations either had a higher limiting value than the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation or no limiting value.

5. Axial shearing of a circular cylinder

Next we consider the axial (telescopic) shearing of a circular cylinder. A hollow cylindrical tube of homogeneous material is bound on its inner radius to a fixed rigid shaft. At the outer radius the tube is subjected to a uniformly distributed axial shearing force. The tube is assumed to be long so that end effects can be ignored, and the deformation is assumed to be axisymmetric.

In cylindrical coordinates, the body initially occupies the region

$$R_{\mathrm{i}} < R < R_{\mathrm{o}}$$

$$-L < Z < L \tag{59}$$

where R_{i} is the inner radius, R_{o} is the outer radius, and $2L$ is the length of the tube. With the assumption of axisymmetry, the physical components of the displacements can be written as

$$u_{\mathrm{r}} = u_{\mathrm{r}}(R)$$

$$u_{\theta} = 0$$

$$u_z = u_z(R) \tag{60}$$

where u_{r}, u_{θ}, and u_z are the radial, circumferential, and axial physical components of the displacement of a point originally at radial location R.

The boundary conditions on the inner radius are

$$u_{\mathrm{r}}(R_{\mathrm{i}}) = 0$$

$$u_z(R_{\mathrm{i}}) = 0 \tag{61}$$

and on the outer radius the dead load boundary conditions can be written as

$$S_{\mathrm{rr}}(R_{\mathrm{o}}) = 0$$

$$S_{z\mathrm{r}}(R_{\mathrm{o}}) = \frac{V}{\pi R_{\mathrm{o}} L} \tag{62}$$

where S_{rr} and $S_{z\mathrm{r}}$ are physical components of the first Piola–Kirchhoff stress and V is the total shear force on the outer surface.

In presenting the results it will be convenient to nondimensionalize the lengths by the inner radius of the tube R_{i} and to nondimensionalize the stresses by the value μ_0. We will consider a tube with

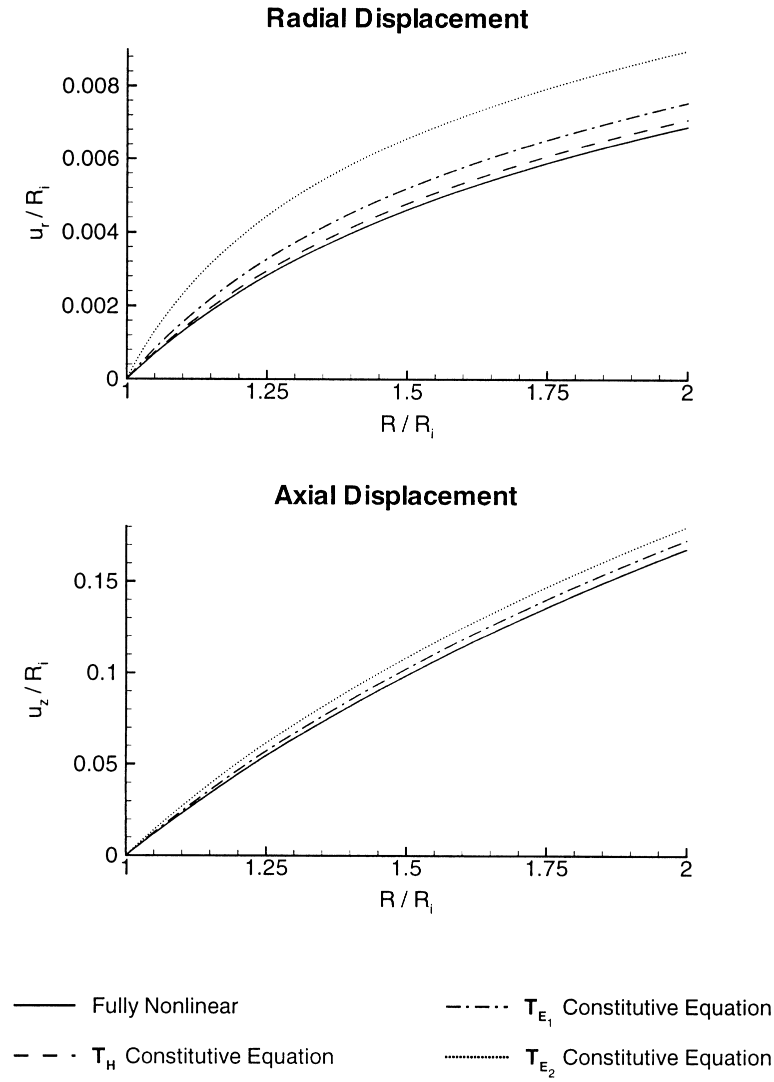

Fig. 4. Displacement for a circular cylinder of generalized Blatz–Ko material with $f = 0$ in axial shear with a total shear force of $V = 1.5\pi\mu_0 R_i L$.

$$\frac{R_o}{R_i} = 2. \tag{63}$$

A general finite element program was developed to solve this problem and the problems that will be presented in Sections 6 and 7.

5.1. Generalized Blatz–Ko material

For this problem, we will discuss the results for the generalized Blatz–Ko material with both $f = 0$ and $f = 1$.

5.1.1. Material parameter $f = 0$

When the shear force is sufficiently small, the strains in the tube will of course be small, and the displacements and stresses calculated with each of the constitutive equations will coincide. As the shear force is increased, the results obtained with the various constitutive equations diverge. For this deformation, the rotation tensor $\mathbf{R} \neq \mathbf{1}$, so the $\mathbf{T_H}$ constitutive equation and the $\mathbf{T_{E_1}}$ constitutive equation will give different solutions.

Because the deformation is not homogeneous, the differences in the solutions obtained with the various constitutive equations will depend on location. This can be seen in Fig. 4, which shows the

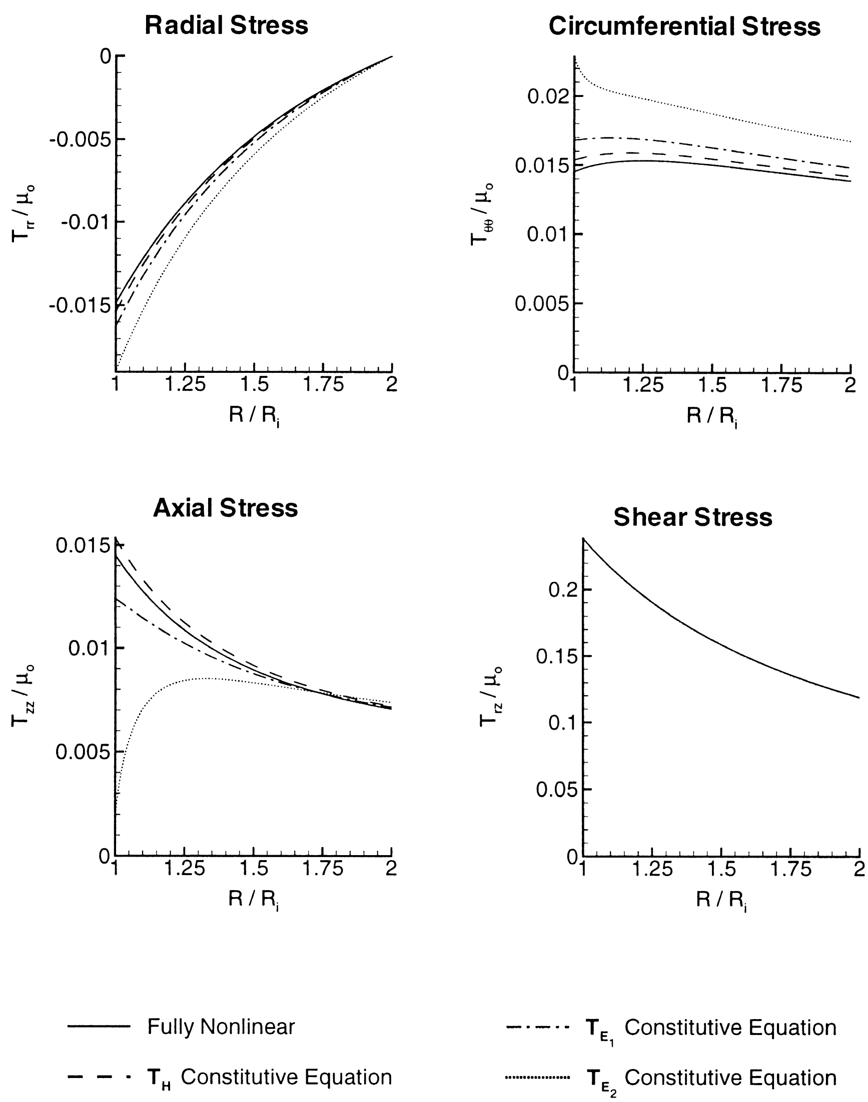

Fig. 5. Cauchy stress for a circular cylinder of generalized Blatz–Ko material with $f = 0$ in axial shear with a total shear force of $V = 1.5\pi\mu_0 R_i L$.

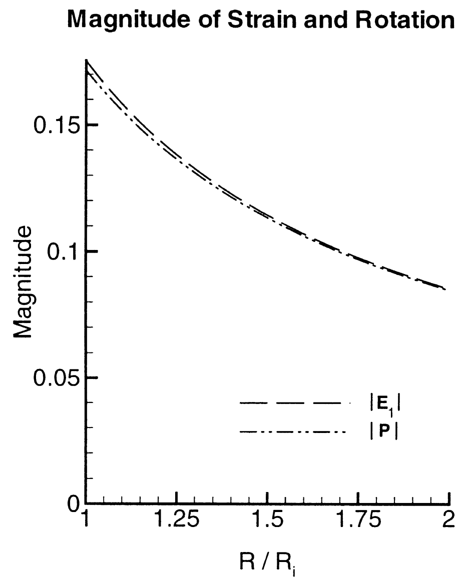

Fig. 6. Magnitude of strain and rotation for a circular cylinder of generalized Blatz–Ko material with $f=0$ in axial shear with a total shear force of $V=1.5\pi\mu_0 R_i L$.

radial and axial components of the displacement for a total shear force of $1.5\pi\mu_0 R_i L$. As expected, since there is a displacement boundary condition on the inner boundary, the largest percent differences in the displacement are at the outer radius. The Cauchy stresses calculated with each of the four constitutive equations are displayed in Fig. 5. The distribution of the magnitudes of $\mathbf{E}_1$ and of $\mathbf{P}$ for this loading are displayed in Fig. 6; the average[6] of $|\mathbf{E}_1|$ is 11.9% and the average value of $|\mathbf{P}|$ (where, recall, $\mathbf{P}=\mathbf{R}-\mathbf{1}$) is 11.8%.

For the bar in simple tension, it was easy to calculate the percent difference for each of the solutions because the deformation was homogeneous. In this problem, however, the deformation is inhomogeneous. Therefore, the percent difference in the displacement and stress components obtained with the second-order constitutive equations will vary with position. To streamline our presentation, we will define a scalar measure of the percent difference. Let $\zeta(\mathbf{X},\beta)$ represent a quantity of interest (such as a component of displacement or stress) at location $\mathbf{X}$ for a value β of the loading parameter (which for this problem is the total shear force V). We will use $\zeta_{\mathrm{nl}}(\mathbf{X},\beta)$ to denote the value of this quantity obtained by using the fully nonlinear constitutive equation and $\zeta_{\mathrm{2nd}}(\mathbf{X},\beta)$ to denote the value of this quantity obtained by using a second-order constitutive equation. Define

$$\frac{\max\limits_{\bar{\mathrm{X}}\in\mathscr{B}} \left|\zeta_{\mathrm{nl}}(\bar{\mathbf{X}},\beta)-\zeta_{\mathrm{2nd}}(\bar{\mathbf{X}},\beta)\right|}{\max\limits_{\mathrm{X}\in\mathscr{B}} \left|\zeta_{\mathrm{nl}}(\mathbf{X},\beta)\right|} \tag{64}$$

[6] The average used here and for the circular shear problem is a radial average, i.e., the magnitude of the strain is calculated at locations equally spaced radially and averaged. It is not a volume average.

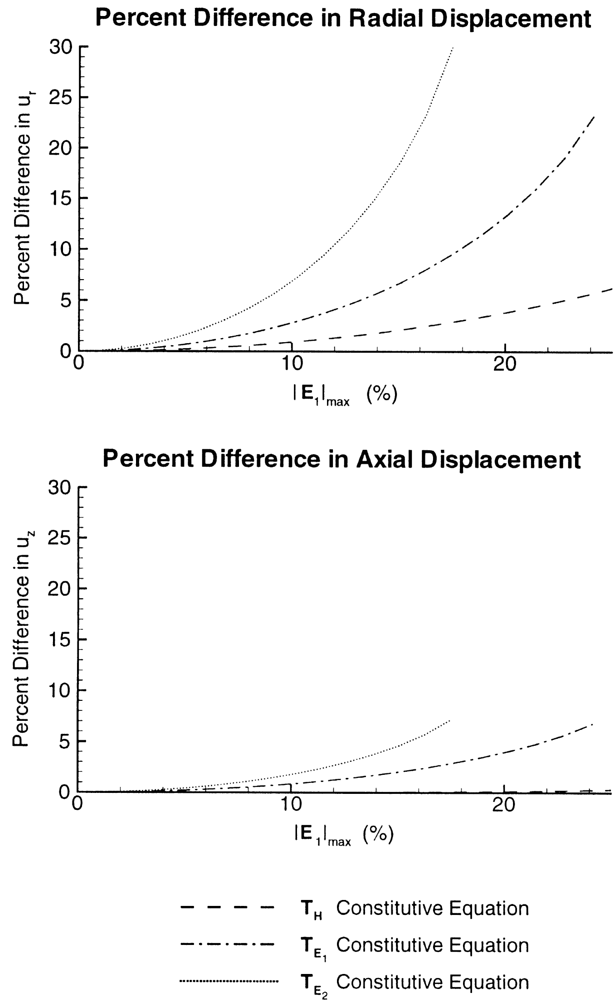

Fig. 7. Percent difference in physical components of displacement for a circular cylinder of generalized Blatz–Ko material with $f = 0$ in axial shear.

as the *percent difference for* ζ. These percent differences will be displayed versus the maximum value of $|\mathbf{E}_1|$ found throughout the body where the value of $|\mathbf{E}_1|$ is determined with the fully nonlinear constitutive equation.

The percent differences in the displacement components are shown in Fig. 7 and the percent differences in the non-zero components of the Cauchy stress are displayed in Fig. 8. The percent differences for the radial–axial shear stress are very small; so small, in fact, that their curves barely rise above the horizontal axis. This feature is unique to this component of the stress since, due to the geometry of the body and loading, this component depends on the nature of the constitutive equation only through the small radial displacement.

Examination of Figs. 7 and 8 establishes that for the problem of axial shear of the generalized Blatz–Ko material with $f = 0$, the $\mathbf{T_H}$ constitutive equation gives the smallest percent difference in the components of the displacement and the stress, and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation gives the largest

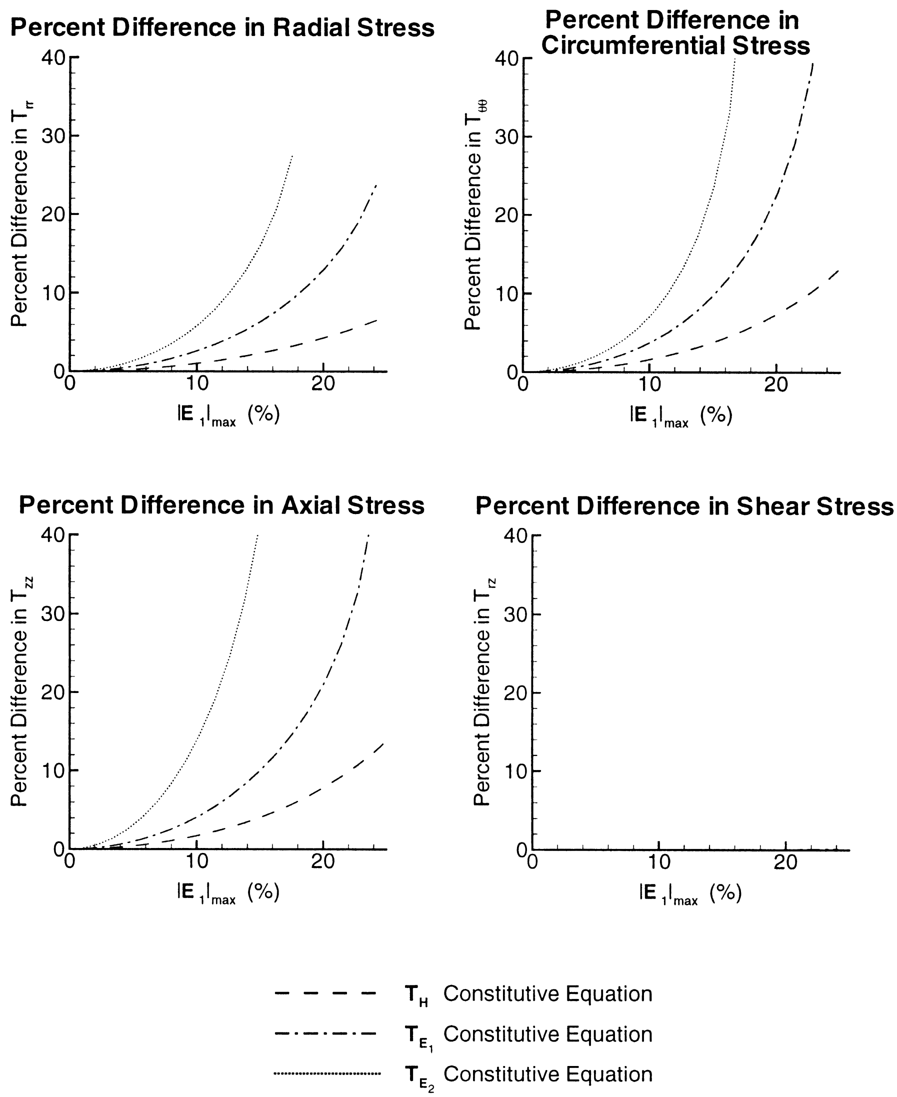

Fig. 8. Percent difference in physical components of Cauchy stress for a circular cylinder of generalized Blatz–Ko material with $f = 0$ in axial shear.

percent differences. For example, when the maximum magnitude of the Biot strain reaches 15%, the percent difference in the calculated axial stress is 4.1% for the $\mathbf{T_H}$ constitutive equation, 10.0% for the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation, and 41.6% for the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation. Alternatively, if the percent differences in the displacement and stress components must be limited to a specified value, the $\mathbf{T_H}$ constitutive equation can be used over the greatest range of strain. For example, if the percent differences are limited to 5%, the $\mathbf{T_H}$ constitutive equation can be used to a strain $|\mathbf{E}_1|_{max}$ of 16.5%, while the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation fails to meet this level of accuracy after a strain of 11.0% is reached, and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation fails to meet this level of accuracy after a strain of only 6.3% has been reached.

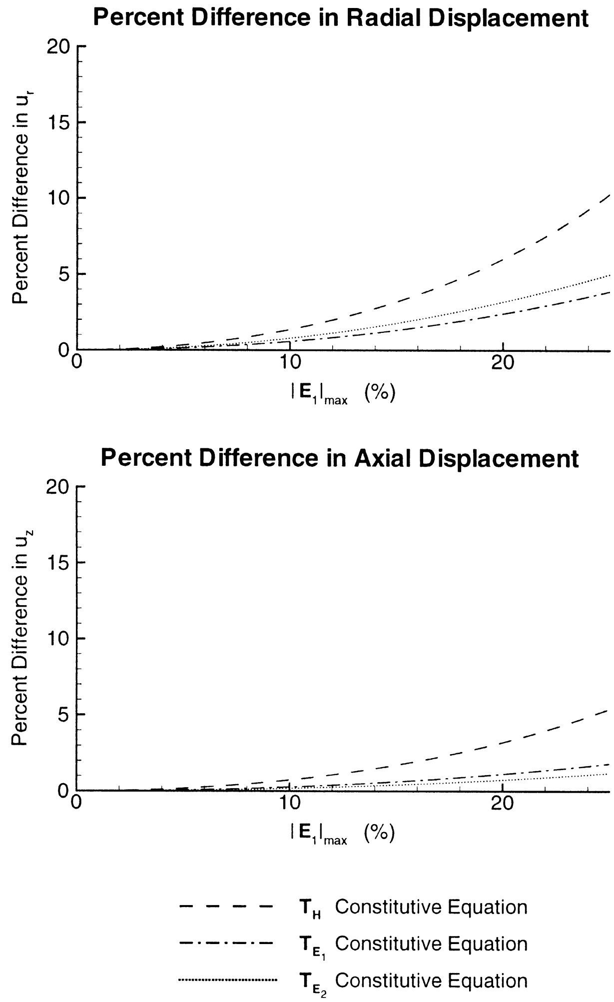

Fig. 9. Percent difference in physical components of displacement for a circular cylinder of harmonic material in axial shear.

For this problem it is not possible to obtain smooth solutions with the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation when $|\mathbf{E}_1|_{\max}$ exceeds approximately 25% or with the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation when $|\mathbf{E}_1|_{\max}$ exceeds approximately 18%. Recall that this loss of physically meaningful solutions was also observed for the second-order constitutive equations with the bar in simple tension. Here, as in that case, the limiting strain is higher for the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation than for the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation.

5.1.2. Material parameter $f = 1$

For the generalized Blatz–Ko material with $f = 1$, the definitions $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{H} = \mathbf{F} - \mathbf{1}$ allow the constitutive equation (25) to be rewritten as

$$\mathbf{T} = \mu_0[-III_{\mathrm{B}}^{(q-1)/2} + III_{\mathrm{B}}^{-1/2}]\mathbf{1} + \mu_0 III_{\mathrm{B}}^{-1/2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}) + \mu_0 III_{\mathrm{B}}^{-1/2}\mathbf{H}\mathbf{H}^{\mathrm{T}}. \tag{65}$$

For a problem in which the deformation is isochoric, this constitutive equation is exactly second-order

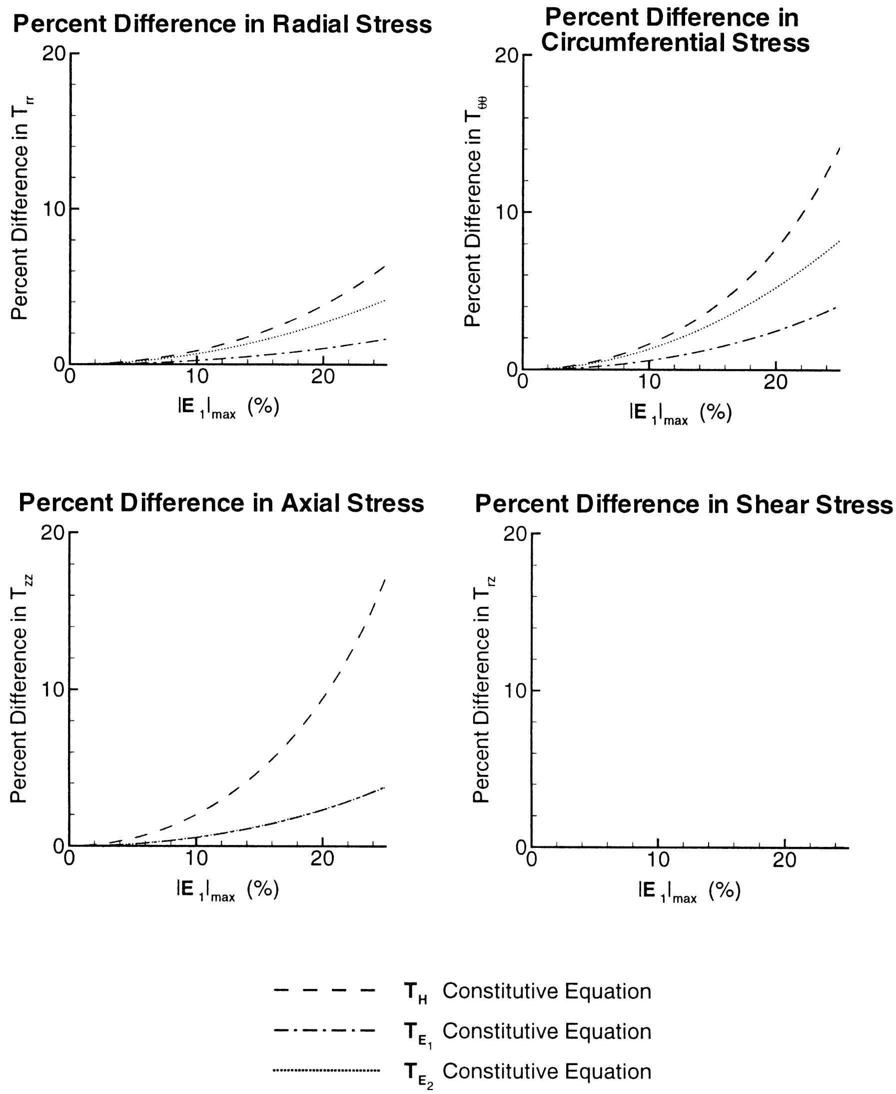

Fig. 10. Percent difference in physical components of Cauchy stress for a circular cylinder of harmonic material in axial shear.

in the displacement gradient **H**. Thus, in such a problem, the $\mathbf{T_H}$ constitutive equation will give the same solution as the fully nonlinear generalized Blatz–Ko constitutive equation[7].

Polignone and Horgan (1992) established that axial shear of the generalized Blatz–Ko material with $f = 1$ gives rise to an isochoric deformation. Thus, for this problem, the $\mathbf{T_H}$ constitutive equation will give the same solution as is obtained with the fully nonlinear constitutive equation. Our calculations produced this expected result.

[7] The generalized Blatz–Ko constitutive equation with $f = 1$ will not be second-order in **H** for a general deformation because III_B must also be expanded in **H** in that case. Therefore, for a general deformation, the $\mathbf{T_H}$ constitutive equation will not give the same solution as the fully nonlinear constitutive equation.

We note that the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives more accurate results than does the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation for this problem, which parallels previous findings. For example, when $|\mathbf{E}_1|_{max}$ reaches 15%, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives a percent difference in the axial displacement of 0.6% and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation gives a percent difference of 1.3%. Because of the special nature of this problem, we will not display the results obtained.

5.2. Harmonic material

The results for axial shear of a cylinder composed of harmonic material are displayed in Figs. 9 and 10 as functions of $|\mathbf{E}_1|_{max}$. Unlike the case of the generalized Blatz–Ko material, for the harmonic material both the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations yield smaller percent differences than does the $\mathbf{T}_{\mathbf{H}}$ constitutive equation. For example, when $|\mathbf{E}_1|_{max}$ reaches 15%, the percent difference in the axial stress for the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation is 1.2%, and the percent difference for the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation is 1.3%, while the percent difference for the $\mathbf{T}_{\mathbf{H}}$ constitutive equation is 4.8%. Of course, this implies that the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation can be used over a larger range than the other two constitutive equations if the percent differences are to remain below a specified level. For example, if the percent differences in each of the components of displacement and stress is to be limited to 5%, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation can be used up to values of $|\mathbf{E}_1|_{max}$ of 27.2% while the $\mathbf{T}_{\mathbf{E}_2}$ and the $\mathbf{T}_{\mathbf{H}}$ constitutive equations can only be used for values of $|\mathbf{E}_1|_{max}$ less than 19.5 and 15.3%, respectively.

5.3. Summary: axial shearing of a circular cylinder

The results obtained from this problem when the size of the strains and rotations become large illustrate the following points:

- As expected, the percent differences in the components of the displacement and stress obtained using the second-order constitutive equations are a function of position for an inhomogeneous deformation.
- The nature of the material can affect the magnitude of the percent differences obtained with the second-order constitutive equations and even the relative performance of the different second-order equations. As for a bar in simple tension, the percent differences obtained for axial shearing of a cylinder using the second-order constitutive equations for the harmonic material are smaller than the percent differences obtained for the generalized Blatz–Ko material. In the axial shearing case, however, the material affects which of the second-order constitutive equations gives the smallest percent differences with the $\mathbf{T}_{\mathbf{H}}$ constitutive equation giving the smallest percent differences for the generalized Blatz–Ko material (in fact, if $f = 1$, the $\mathbf{T}_{\mathbf{H}}$ constitutive equation gives zero percent difference) and the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation giving the smallest percent differences for the harmonic material.
- Constitutive equations that are second-order in a certain measure of the strain may yield smaller percent differences for some components of the displacement and the stress than the other second-order constitutive equations while yielding larger percent differences for other components. For example, for the axial shearing of a cylinder composed of harmonic material, the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation gives smaller percent differences for the axial deformation than either of the other two second-order constitutive equations; however, when taking all components of the displacements and stresses into account, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives results that are closest to those from the fully nonlinear constitutive equation.
- As previously noted for the bar in simple tension, the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation fails to have physically meaningful solutions at lower levels of strain than either the $\mathbf{T}_{\mathbf{E}_1}$ or the $\mathbf{T}_{\mathbf{H}}$ constitutive equations.

6. Circular shear of a circular cylinder

We now turn to the circular, or circumferential, shear of a hollow cylindrical tube of homogeneous material. The material is bound at its inner radius to a fixed rigid shaft and at its outer radius to a rigid concentric shell which is twisted about the common axis through a prescribed angle. The tube is assumed to be long enough to ensure that end effects can be neglected, and the deformation is assumed to be axisymmetric.

The body occupies the region defined by

$$R_{\mathrm{i}} < R < R_{\mathrm{o}}$$

$$-L < Z < L \tag{66}$$

where R_{i} is the inner radius, R_{o} is the outer radius, and $2L$ is the length of the tube.

The physical components of the displacement are assumed to have the form

$$u_{\mathrm{r}} = u_{\mathrm{r}}(R)$$

$$u_{\theta} = u_{\theta}(R)$$

$$u_z = 0 \tag{67}$$

where u_{r}, u_{θ} and u_z are the radial, circumferential, and axial physical components of the displacement of a point originally located at radius R.

The boundary conditions on the inner surface are

$$u_{\mathrm{r}}(R_{\mathrm{i}}) = 0$$

$$u_{\theta}(R_{\mathrm{i}}) = 0 \tag{68}$$

and on the outer surface the displacement boundary conditions are

$$u_{\mathrm{r}}(R_{\mathrm{o}}) = R_{\mathrm{o}}(1 - \sin\phi)$$

$$u_{\theta}(R_{\mathrm{o}}) = R_{\mathrm{o}}\cos\phi \tag{69}$$

where ϕ is the prescribed rotation of the outer cylinder.

It is convenient to nondimensionalize the lengths by the inner radius of the tube R_{i} and the stresses by μ_0. We will consider a tube with

$$\frac{R_{\mathrm{o}}}{R_{\mathrm{i}}} = 2. \tag{70}$$

6.1. Generalized Blatz–Ko material

6.1.1. Material parameter $f = 1$

Fig. 11 shows the displacements obtained using the fully nonlinear and the three second-order constitutive equations for the generalized Blatz–Ko material with an outer shell rotation of 0.20 rad.

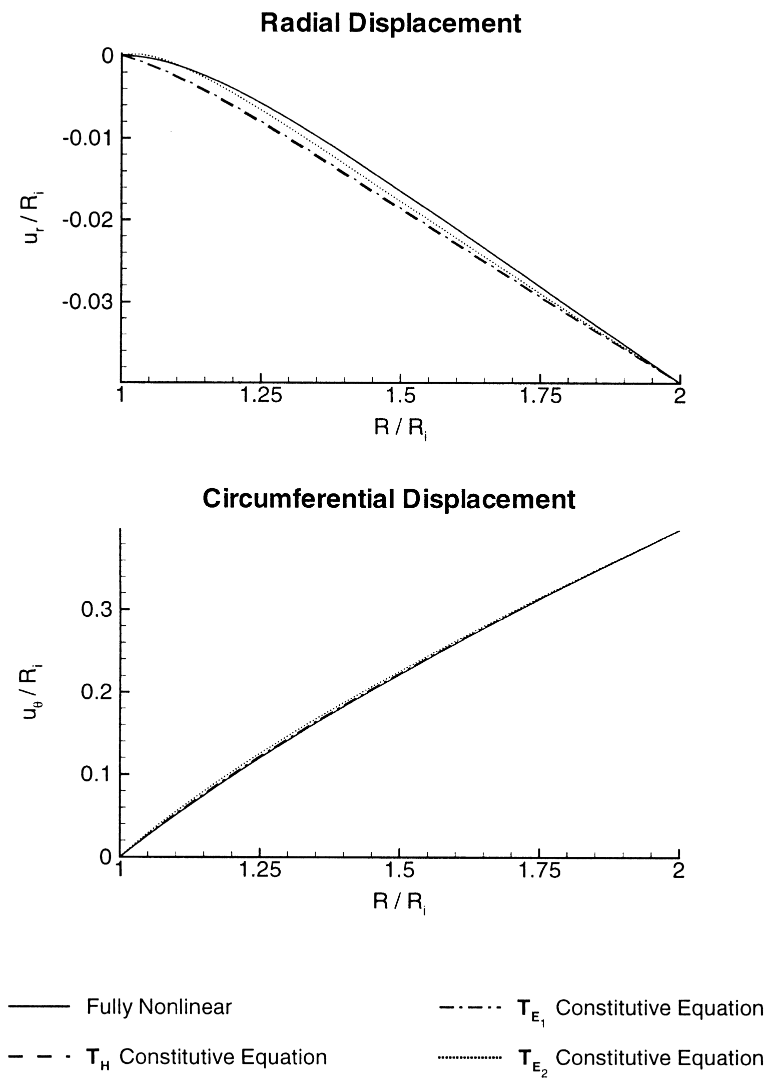

Fig. 11. Displacement for a circular cylinder of generalized Blatz–Ko material with $f = 1$ in circular shear with an outer shell rotation of 0.20 rad.

The non-zero physical components of the Cauchy stress obtained for this problem are shown in Fig. 12. As expected, the percent differences in the stress components are significantly larger than the percent differences in the displacement components because the displacements at both the inner and the outer radii are prescribed.

For $\phi = 0.20$ rad, the average[8] value of $|\mathbf{E}_1|$ obtained using the fully nonlinear constitutive equation is 18.9% and the average value of $|\mathbf{P}|$ is 37.4%. The distributions of $|\mathbf{E}_1|$ and $|\mathbf{P}|$ in the cylinder are shown in Fig. 13. By comparing this figure to Fig. 6, we see that the average value of the rotation is higher for

[8] The average used here is a radial average.

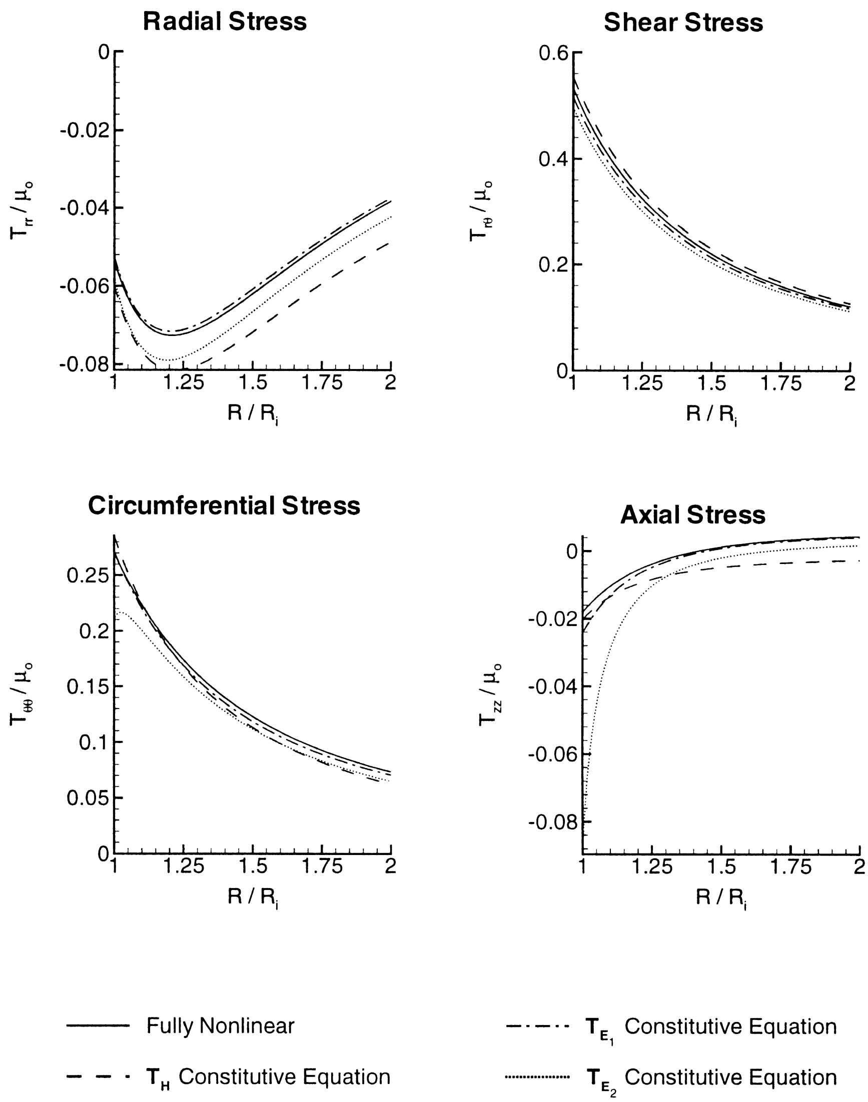

Fig. 12. Cauchy stress for a circular cylinder of generalized Blatz–Ko material with $f = 1$ in circular shear with an outer shell rotation of 0.20 rad.

this problem than it is in the axial shearing problem even though the maximum rotation in the two problems is approximately the same.

The percent differences in the physical components of the displacement and the stress are shown in Figs. 14 and 15, respectively. Note that in all the cases (except for slightly more accurate results for the circumferential component of the displacement with the $\mathbf{T_H}$ constitutive equation) the percent differences associated with the solutions from the $\mathbf{T_{E_1}}$ constitutive equation are smaller than the percent differences from the other two second-order constitutive equations. For example, when the maximum value of $|\mathbf{E}_1|$ is 15%, the percent difference in the radial stress is 0.3% for the $\mathbf{T_{E_1}}$ constitutive equation, 0.8% for the $\mathbf{T_{E_2}}$ constitutive equation, and 2.3% for the $\mathbf{T_H}$ constitutive equation. With the criteria that the percent difference in all components of the displacement and stress remain below a specified threshold, Figs. 14

and 15 show that the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations are useful over a larger range of strain than is the $\mathbf{T}_\mathbf{H}$ constitutive equation. For example, for a maximum acceptable percent difference of 5%, the allowable range of $|\mathbf{E}_1|$ for the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation goes to 15.2%, and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation can be used up to a strain of 13.2%, whereas the $\mathbf{T}_\mathbf{H}$ constitutive equation produces 5% differences at strains of 6.6%.

6.1.2. Material parameter f = 3/4

It was shown by Haughton (1993) and Polignone and Horgan (1994) that for circular shearing of the Blatz–Ko material with $f = 3/4$, the deformation is isochoric. So, by analogy with the problem of axial shearing of the Blatz–Ko material with $f = 1$, it might be expected that in this problem the $\mathbf{T}_\mathbf{H}$ constitutive equation will also be exact. This is not the case, however, because the term

$$-\frac{\mu_0(1-f)}{III_\mathrm{B}^{1/2}}\mathbf{B}^{-1} \tag{71}$$

which appears in the constitutive equation when $f \neq 1$, is not second-order in $\mathbf{H}$ even if $\mathrm{III}_\mathbf{B} = 1$.

In fact, the percent differences for the circular shearing problem with the Blatz–Ko material with $f = 3/4$ are similar to those obtained for $f = 1$. For example, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives a percent difference in the radial stress of 0.4% for $|\mathbf{E}_1|_{\max} = 15\%$, the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation gives a percent difference in the radial stress of 0.8%, and the $\mathbf{T}_\mathbf{H}$ constitutive equation gives a percent difference of 2.0%.

6.1.3. Material parameter f = 0

As was shown by Knowles and Sternberg (1975b) and illustrated by Wineman and Waldron (1995), the equations of elastostatics for this problem lose ellipticity for sufficiently large deformations. This causes numerical difficulties in obtaining a solution. It is possible, however, to obtain some results, and

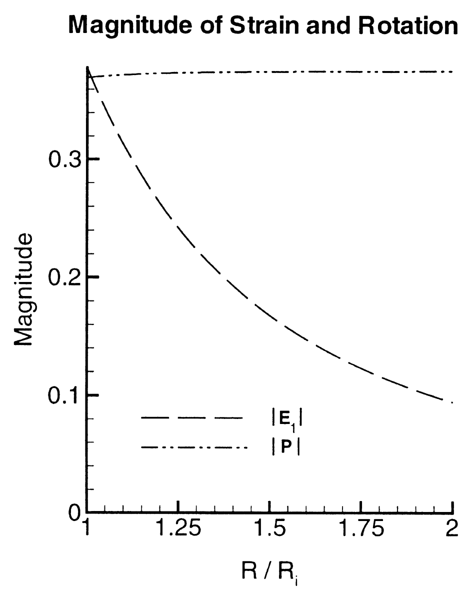

Fig. 13. Magnitude of strain and rotation for a circular cylinder of generalized Blatz–Ko material with $f = 1$ in circular shear with an outer shell rotation of 0.20 rad.

these results show trends similar to results obtained for other values of the material parameter *f*. For example, at maximum of $|\mathbf{E}_1|$ of about 9.5%, the percent difference in the radial stress calculated using the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation is about 0.1%, using the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation the percent difference is about 0.6%, and using the $\mathbf{T}_\mathbf{H}$ constitutive equation the percent difference is about 0.8%. While these percent differences are quite small, it can be expected that the relative performance of the three second-order constitutive equations would continue to higher levels of $|\mathbf{E}_1|$ since, for all the problems studied, in only a very few cases is there a difference between the relative performance of the second-order constitutive equations for small values of $|\mathbf{E}_1|$ and larger values of $|\mathbf{E}_1|$.

6.2. Harmonic material

The percent differences in the displacement components and the non-zero components of the Cauchy stress obtained with the second-order constitutive equations for the harmonic material are displayed in Figs. 16 and 17, respectively. For all components of the displacement and stress, the $\mathbf{T}_\mathbf{H}$ constitutive equation yields larger percent differences than does the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation. For example, when

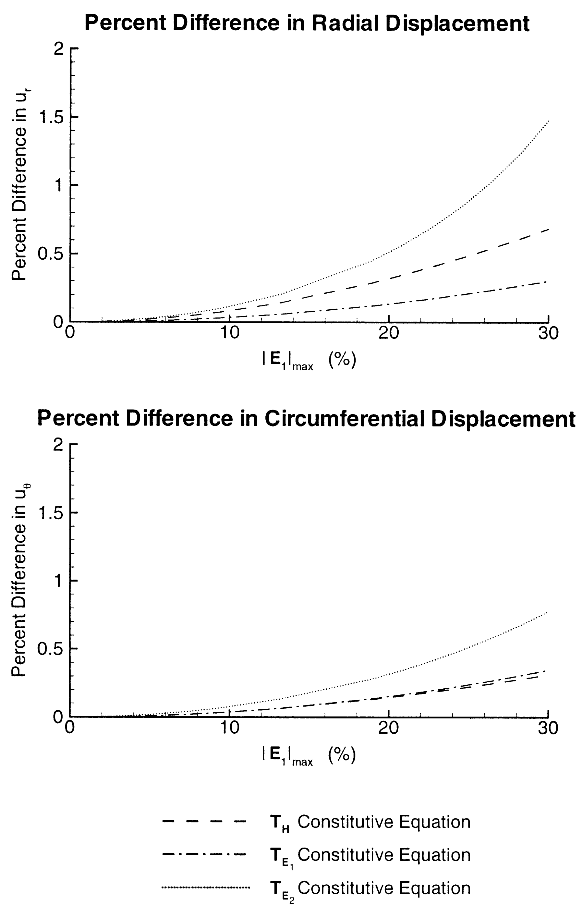

Fig. 14. Percent difference in physical components of displacement for a circular cylinder of generalized Blatz–Ko material with $f = 1$ in circular shear.

$|\mathbf{E}_1|_{max}$ reaches 15%, the percent difference in the radial stress is 3.5% for the $\mathbf{T_H}$ constitutive equation, but only 0.3% for the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation. On the other hand, for some of the components the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation yields better results than the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation while for other it yields worse results. If all of the components of the displacement and stress are restricted to have percent differences below some specified value, then the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation has the greatest range of applicability. For example, if the percent differences are restricted to 5%, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation can be used up to values of $|\mathbf{E}_1|_{max}$ of 48.2% while the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation can only be used up to 18.5% strain and the $\mathbf{T_H}$ constitutive equation only up to 18.1% strain.

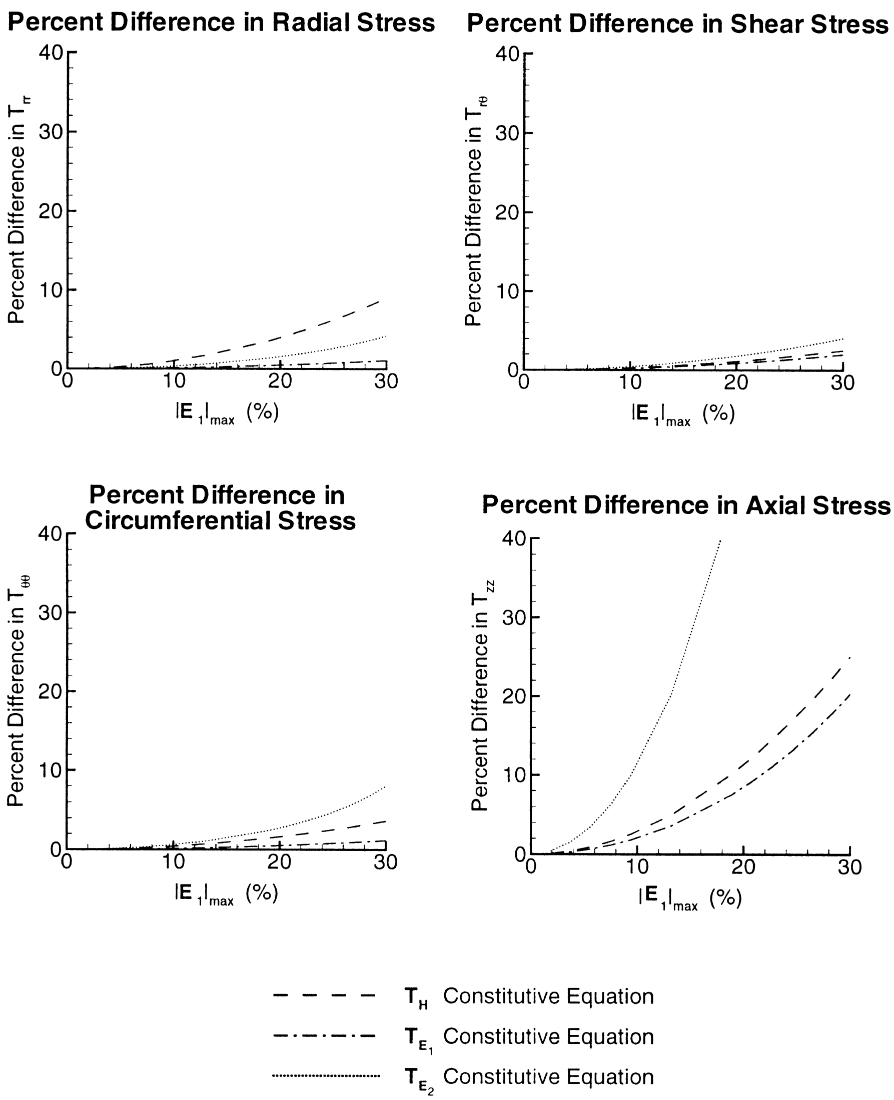

Fig. 15. Percent difference in physical components of Cauchy stress for a circular cylinder of generalized Blatz–Ko material with $f = 1$ in circular shear.

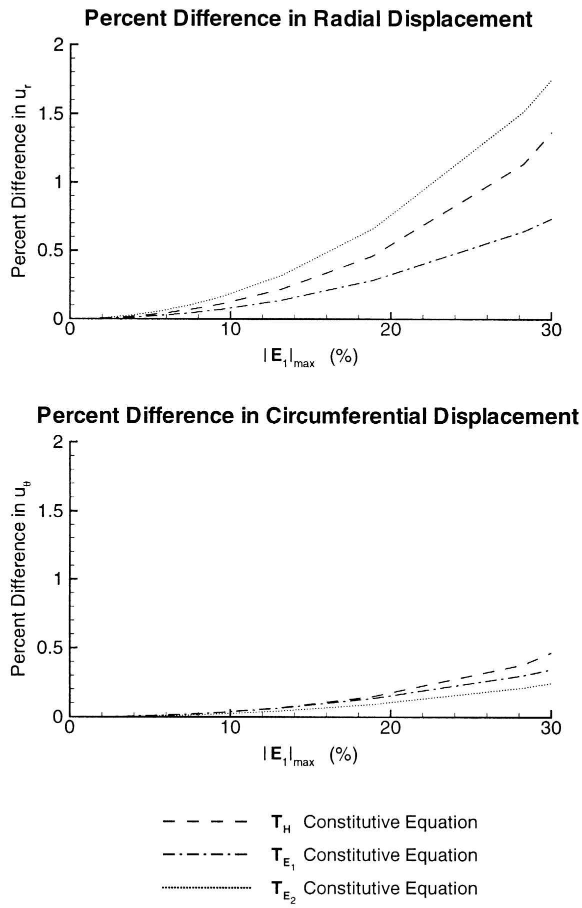

Fig. 16. Percent difference in physical components of displacement for a circular cylinder of harmonic material in circular shear.

6.3. Summary: circular shear of a circular cylinder

The problem of circular shear of a cylinder illustrates that:

- As expected, the presence of a substantial rotation can dramatically reduce the accuracy of the solution obtained with the $\mathbf{T_H}$ constitutive equation. This effect can be large enough to overcome any advantage that the $\mathbf{T_H}$ constitutive equation may have in describing a particular material. For example, even though the $\mathbf{T_H}$ constitutive equation gave the smallest percent differences for the axial shearing of a cylinder of a generalized Blatz–Ko material, for the circular shearing of a cylinder of a generalized Blatz–Ko material the $\mathbf{T_H}$ constitutive equation gave the largest percent differences. This is, of course, a consequence of the fact that the $\mathbf{T_H}$ constitutive equation is not objective and the rigid body rotations are larger in circular shearing than axial shearing. It should be noted, however, that the difference in the magnitude of the rotations for the two problems is only moderate, but this difference has a large impact on the accuracy of the solution.
- As with the other examples, the nature of the material can affect the percent differences in the solutions obtained using a second-order constitutive equation. For example, for the generalized Blatz–

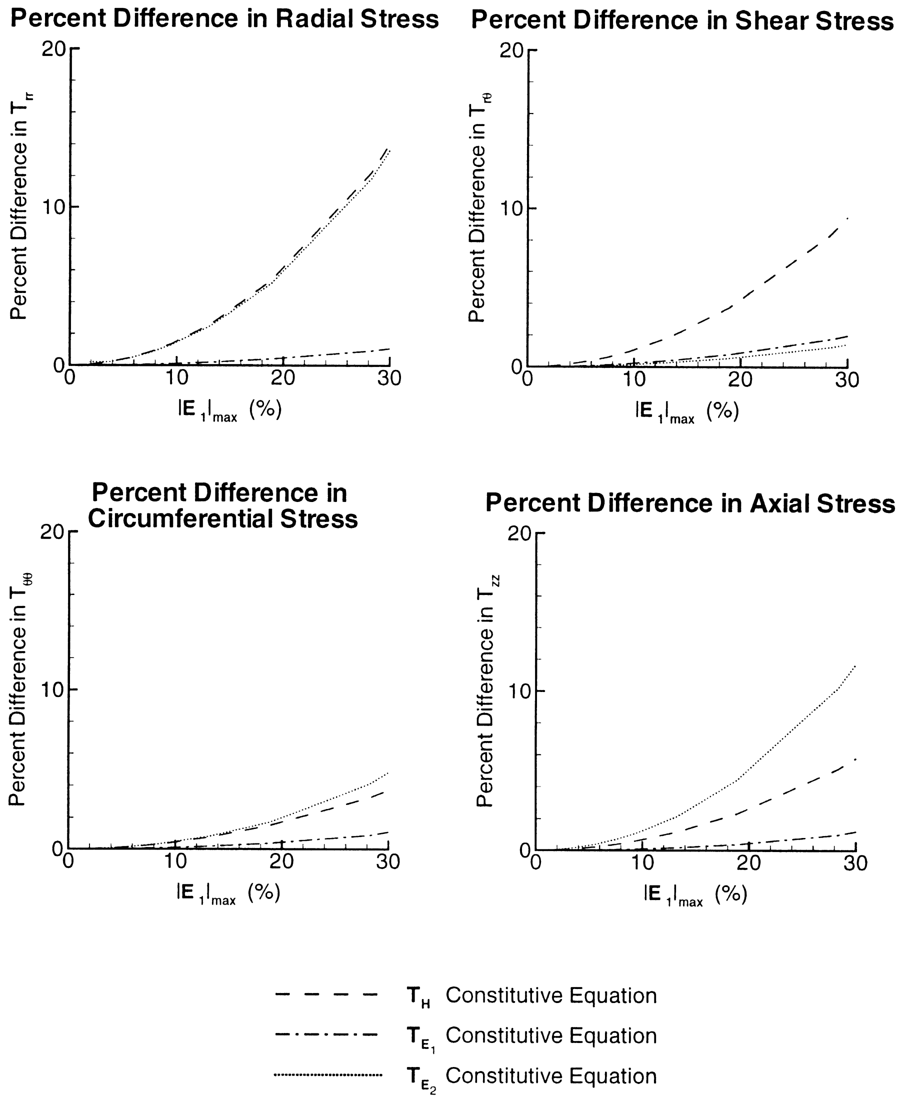

Fig. 17. Percent difference in physical components of Cauchy stress for a circular cylinder of harmonic material in circular shear.

Ko material the percent differences are larger than the percent differences for the harmonic materials in both axial shearing and circular shearing.

7. Bending of a cantilever beam

The final problem we consider is bending of a cantilever beam. Since the rotations in a beam are large, we expect the $\mathbf{T_H}$ constitutive equation to produce solutions with large percent differences. A rectangular beam of homogeneous material is cantilevered at one end and the bottom edge of the other

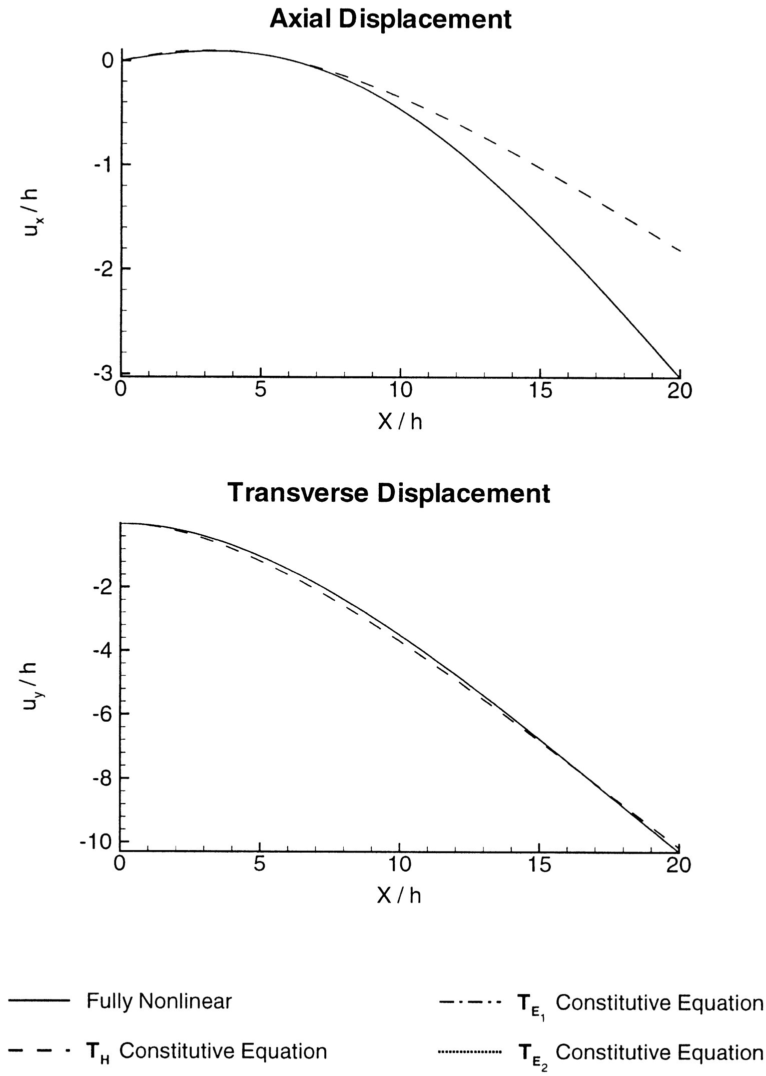

Fig. 18. Displacement for a cantilever beam of generalized Blatz–Ko material with $f = 0$ in bending with tip displacement $d/h = 10$.

end is displaced transversely to its axis by a prescribed amount. We assume the beam is sufficiently jthick in the direction perpendicular to the plane of the beam that a plane strain condition can be assumed[9].

The body initially occupies a region given by

[9] Results were also obtained for a beam assuming a plane stress condition and for a beam with a prescribed thickness. These results are very similar to the results obtained using the plane strain assumption, so because they yield no additional insights, they will not be presented here.

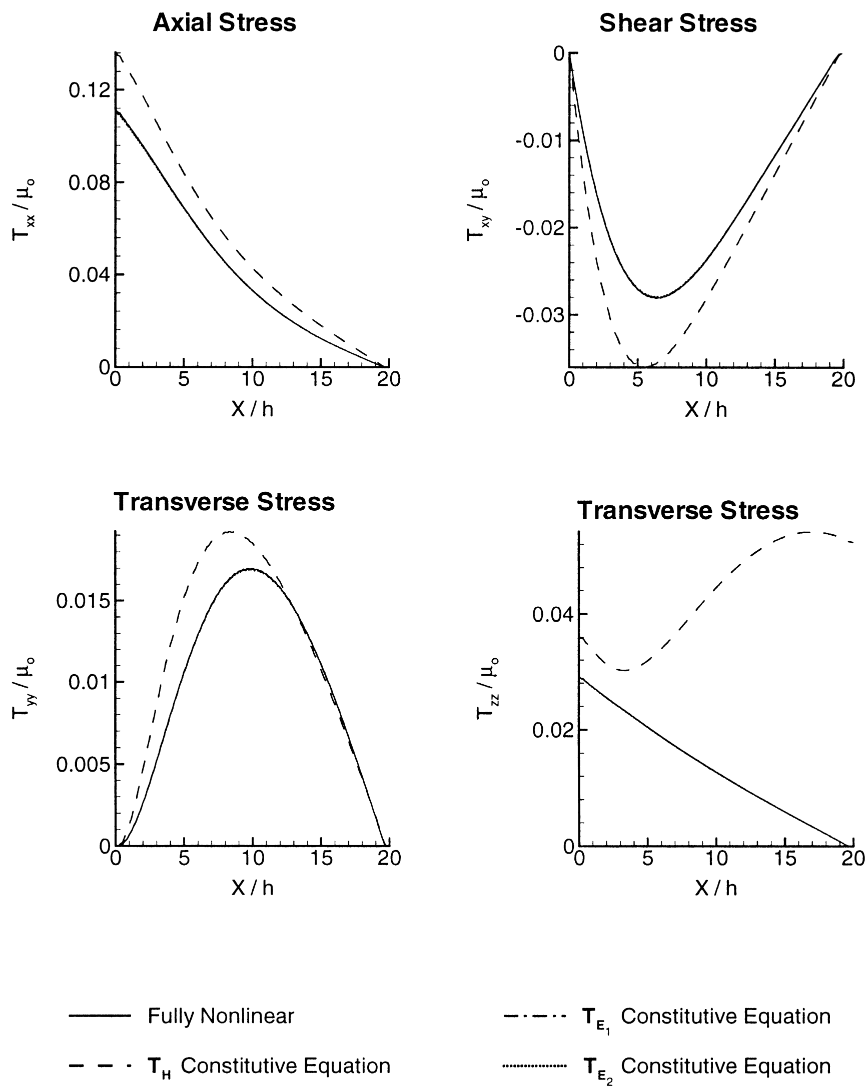

Fig. 19. Cauchy stress for a cantilever beam of generalized Blatz–Ko material with $f = 0$ in bending with tip displacement $d/h = 10$.

$$\begin{aligned} &0 < X < L \\ &0 < Y < h \\ &0 < Z < w \end{aligned} \tag{72}$$

where L is the length of the beam, h the height, and w the width.

Under the plane strain assumption the components of the displacement can be written as

$$\begin{aligned} &u_x = u_x(X, Y) \\ &u_y = u_y(X, Y) \\ &u_z = 0 \end{aligned} \tag{73}$$

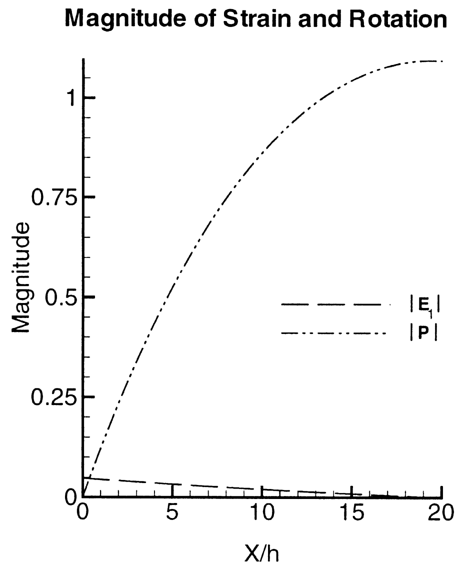

Fig. 20. Magnitude of strain and rotation for a cantilever beam of generalized Blatz–Ko material with $f = 0$ in bending with tip displacement $d/h = 10$.

where u_x, u_y, and u_z are the axial, transverse, and out-of-plane displacements of a point originally at the location defined by (X, Y, Z).

The cantilever end is modeled by the displacement boundary conditions[10]

$$u_x(0, Y) = 0, \quad u_y(0, 0) = 0. \tag{74}$$

The displacement at the free end is prescribed by the boundary condition

$$u_y(L, 0) = -d \tag{75}$$

where d is the amount of the displacement. The remaining boundary conditions all specify that the tractions are zero.

In presenting the results, we will nondimensionalize the lengths by the height h and the stresses by μ_0. We will consider a beam with

$$\frac{L}{h} = 20. \tag{76}$$

7.1. Generalized Blatz–Ko material with $f = 0$

Fig. 18 shows the axial and transverse[11] components of the displacement along the upper surface for a

[10] The second of these boundary conditions creates a stress concentration in the lower, left-hand corner; however, an examination of this stress concentration, including a convergence analysis, showed that this stress concentration has little effect on the results presented in this paper.

[11] The transverse displacements shown in Fig. 18 at the original axial location of 20 do not equal the prescribed displacement because this figure shows the transverse displacement at the top of the beam and the prescribed displacement is applied at the bottom of the beam. The difference between the two is due to rotations and strain of the beam in the transverse direction.

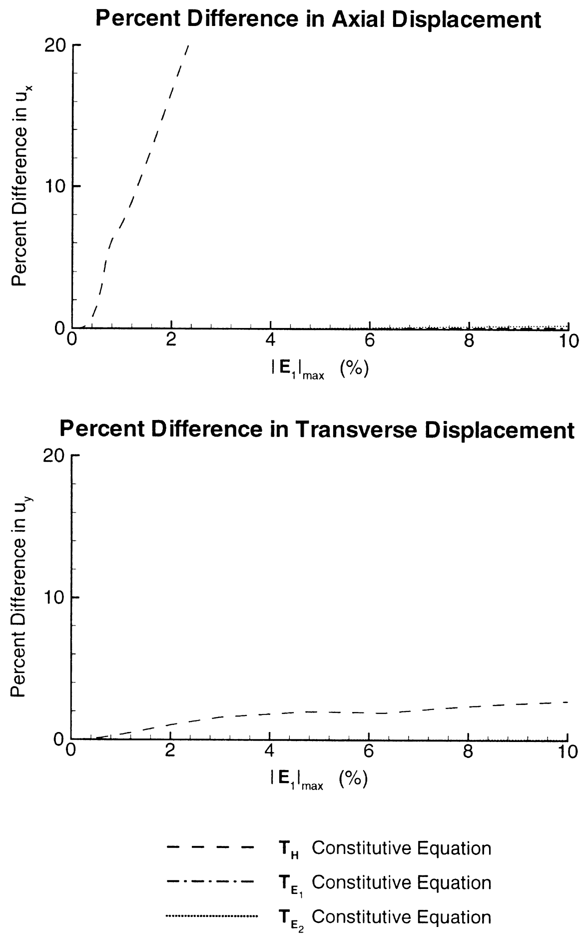

Fig. 21. Percent difference in components of displacement for a cantilever beam of generalized Blatz–Ko material with $f = 0$ in bending.

nondimensionalized free end displacement, d/h, of 10. In this figure the results for the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations cannot be distinguished from the results for the fully nonlinear constitutive equation. Fig. 19 shows the non-zero components of the Cauchy stresses along the top of the beam. Here it is also difficult to distinguish the results for the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations from the fully nonlinear results. Note that the average[12] magnitude of the strain obtained using the fully nonlinear constitutive equation in this case is 2.2% and the average magnitude of **P** is about 75.7%. Fig. 20 shows the magnitude of the rotations and the strains along the top of the beam. Once can see that the rotations for the cantilever beam problem are significantly larger than for either the axial shear or the circular shear problem by comparing Fig. 20 to Figs. 6 and 13.

The percent differences for the displacement are shown in Fig. 21. The percent differences in the axial displacement for the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations are so small that they cannot be

[12] This average is an average over the top of the beam, i.e., the magnitude is calculated at a number of points evenly distributed over the top of the beam and averaged.

distinguished from the horizontal axis. As expected, because of the boundary conditions on the transverse displacement, the percent differences in the axial components of the displacement are much larger than in the transverse components of the displacements. The percent differences in the non-zero components of the Cauchy stresses along the top of the beam are shown in Fig. 22.

By examining these figures, it can be seen that for both the displacements and the stresses, the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations give results with much smaller percent differences than the $\mathbf{T}_{\mathbf{H}}$ constitutive equation, and in particular the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation produces the smallest percent differences. For example, if each of the components of the displacement and stress are to be limited to

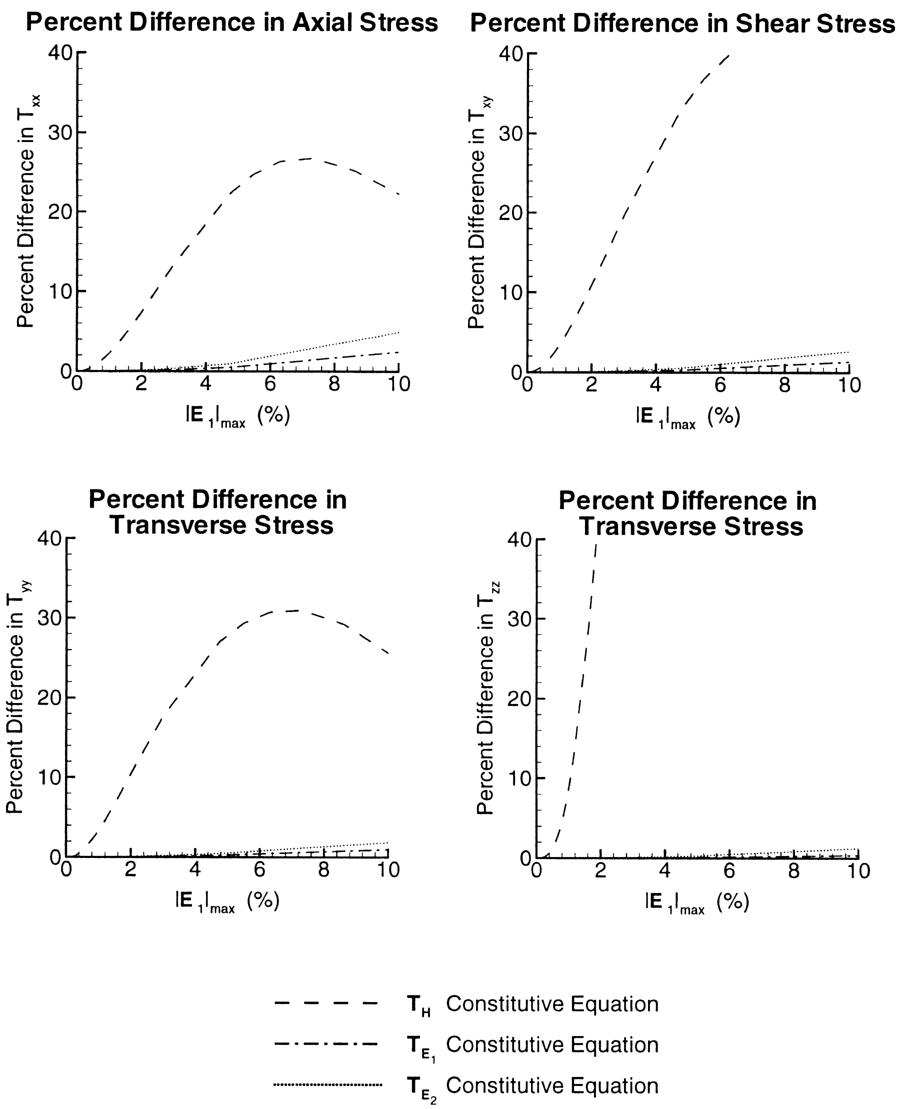

Fig. 22. Percent difference in components of Cauchy stress for a cantilever beam of generalized Blatz–Ko material with $f = 0$ in bending.

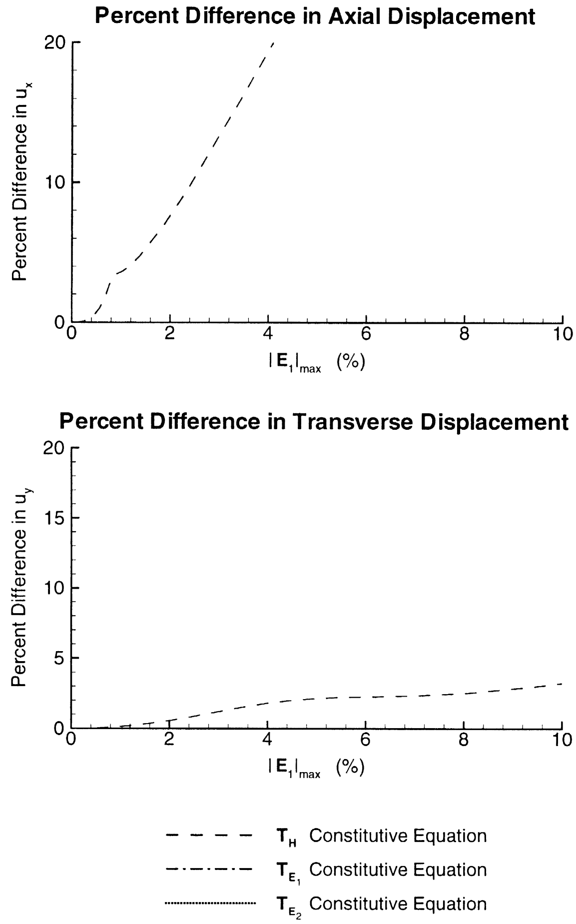

Fig. 23. Percent difference in components of displacement for a cantilever beam of harmonic material in bending.

percent differences of less than 5%, the $\mathbf{T_H}$ constitutive equation can only be used to a maximum value of $|\mathbf{E}_1|$ of 0.7%. On the other hand, the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations can be used to a maximum value of $|\mathbf{E}_1|$ of over 10%. That the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives slightly better results than the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation can be seen by noting that if the percent differences are limited to be less than 1%, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation can be used up to a maximum value of $|\mathbf{E}_1|$ of 6.1% while the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation can be used to a maximum value of only 4.8%.

7.2. Harmonic material

The percent differences obtained in the components of the displacement and stress using the second-order constitutive equations for the cantilever beam made of harmonic material are shown in Figs. 23 and 24, respectively. Clearly, for the components of both the displacement and the stress, the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations both give much better results than the $\mathbf{T}_\mathbf{H}$ constitutive equation. For example, if the percent differences in all the components of the stresses and the displacements are limited to 5%, then the $\mathbf{T}_\mathbf{H}$ constitutive equation can only by used up to a maximum value of $|\mathbf{E}_1|$ of only 0.6% while both the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations can be used up to values of $|\mathbf{E}_1|$ of over 10%. Even if the percent differences are limited to 1%, the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations can still be used up to values of $|\mathbf{E}_1|$ of over 10% which corresponds to a free end deflection, *d*, of over 15 times the height of the beam. Recall that the original length of the beam is only 20 times the height of the beam;

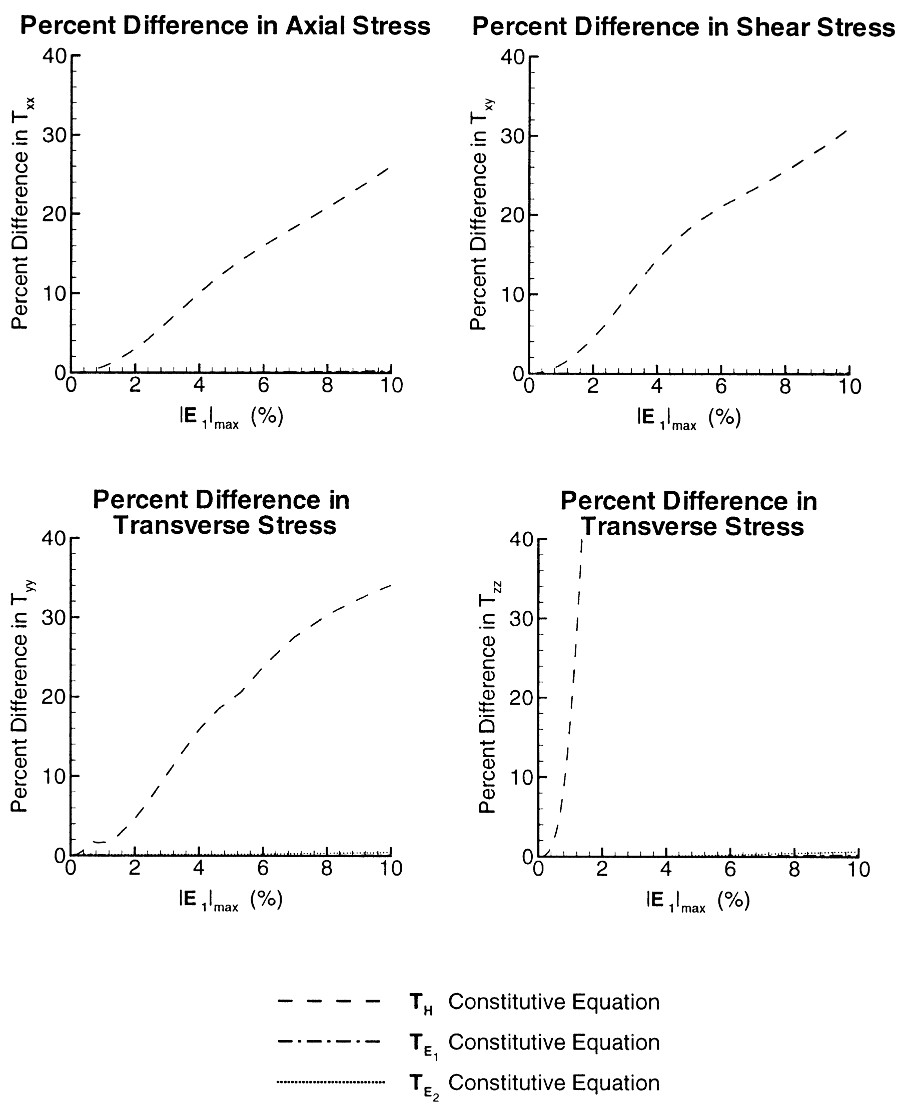

Fig. 24. Percent difference in components of Cauchy stress for a cantilever beam of harmonic material in bending.

this suggest that these constitutive equations are useful even for a severely deformed beam. If all the components are taken into account, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives slightly better results than the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation although for some of the components the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation gives better results than the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation.

7.3. Summary: bending of a cantilever beam

The cantilever beam problem illustrates two main points:

- As expected, when the rotation in a body is large, the $\mathbf{T}_{\mathbf{H}}$ constitutive equation is unusable.
- The $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations, and particularly the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation, can be extremely good approximations to the fully nonlinear constitutive equations for problems where the deformations are dominated by the rotation rather than by the strain.

8. Summary and discussion of results

In this paper we examined the performance of several second-order constitutive equations in four different boundary value problems over a range of strains and rotations. Our goal was to gain some understanding of the circumstances under which such second-order constitutive equations can be used in lieu of fully nonlinear constitutive equations.

The results presented in Sections 4–7 illustrate that there are two factors that play a major role in determining which second-order constitutive equation provides the best approximation to the fully nonlinear constitutive equation: the mechanical nature of the material and the class of deformations that is to be modeled. The nature of the material (meaning here the nature of the fully nonlinear constitutive equation) has a primary effect on the accuracy of the second-order constitutive equations. In our examples, solutions obtained with the second-order approximations to the harmonic material achieved greater accuracy than did the solutions calculated with the second-order approximations to the Blatz–Ko material in every problem but one[13]. In addition to influencing the overall level of accuracy, the nature of the material affects the relative performance of the second-order constitutive equations. This is made evident by comparing the $\mathbf{T}_{\mathbf{H}}$ and the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations for the two materials. For the Blatz–Ko material, as long as the rotations are small, the $\mathbf{T}_{\mathbf{H}}$ constitutive equation gives greater accuracy than does the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation. The consummate example is found in the problem of the axial shear of a cylinder for the Blatz–Ko material with the parameter $f = 1$; in this case the $\mathbf{T}_{\mathbf{H}}$ constitutive equation is exact because the deformation is isochoric. In contrast, for the harmonic material the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives equally as good or better results than does the $\mathbf{T}_{\mathbf{H}}$ constitutive equation in every problem. The impact of the material on the relative performance of the second-order constitutive equations can also be seen by comparing the results for the $\mathbf{T}_{\mathbf{E}_1}$ and the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations. For the Blatz–Ko material, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives more accurate solutions than does the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation for all of the components of the displacement and stress, but for the harmonic material the results obtained with the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation are more accurate for about half of the components and less accurate for the other components. However, we note that at any given magnitude of strain, the largest percent difference of the $\mathbf{T}_{\mathbf{E}_1}$ solution is less than

[13] That exception is the bending of a beam with the $\mathbf{T}_{\mathbf{H}}$ constitutive equation. In that problem the rotations are the dominating factor.

the largest percent difference of the $\mathbf{T}_{\mathbf{E}_2}$ solution. So the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation gives a more accurate solution overall.

The second factor that must be considered when choosing a second-order constitutive equation is the class of deformations that is to be modeled. If it is known in advance that the rotations in a particular problem will be small, the ease of implementing the $\mathbf{T}_{\mathbf{H}}$ constitutive equation may make it attractive to use. However, as the examples in this paper illustrate, if the deformation produces moderate to large rotations, the constitutive equation must be frame indifferent, so the $\mathbf{T}_{\mathbf{H}}$ constitutive equation should not be used. This consideration can override any advantage that the $\mathbf{T}_{\mathbf{H}}$ constitutive equation may seem to have for describing a particular material. For example, in the axial shearing problem where the rotations are small, the $\mathbf{T}_{\mathbf{H}}$ constitutive equation produces the most accurate solution for the generalized Blatz–Ko material. On the other hand, for the circular shearing problem, where the rotations are only slightly larger, the $\mathbf{T}_{\mathbf{H}}$ constitutive equation gives a less accurate solution than does the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation. In bending of a cantilever beam, where the rotations are very significant, the results obtained with the $\mathbf{T}_{\mathbf{H}}$ constitutive equation are as much as an order-of-magnitude less accurate than the results obtained with either the $\mathbf{T}_{\mathbf{E}_1}$ or the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equations.

One additional factor that affects the range of strains for which a second-order constitutive equation can be applied is the loss of smooth solutions to the governing equations. Recall in the problem of a bar in simple tension, that for extensions above a certain critical level it is not possible to obtain a real solution with any of the second-order theories, even though solutions can be obtained for any strain with the fully nonlinear constitutive equation. In the other problems as well, it was not possible to obtain smooth solutions to the governing differential equations under certain circumstances. The critical strain at which it is no longer possible to obtain a smooth solution is smaller for the $\mathbf{T}_{\mathbf{E}_2}$ constitutive equation than for either the $\mathbf{T}_{\mathbf{H}}$ or the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equations. It should be noted, however, that the solutions obtained with the second-order constitutive equations differ substantially from the solution provided by the fully nonlinear constitutive equation long before this critical strain is reached.

In summary, the constitutive equation which is second order in the displacement gradient is not frame indifferent, and therefore, leads to very large percent differences in problems with even moderate rotations. The constitutive equation which is second order in the Green strain is significantly less accurate than is the constitutive equation which is second order in the Biot strain. Thus, of the constitutive equations considered, the constitutive equation which is second order in the Biot strain provides the best approximation of the mechanical behavior of the nonlinear Blatz–Ko and harmonic materials over a large range of strains and rotations. With the criterion that each component of the predicted displacement and the predicted stress be accurate to within 5% of the value determined with the fully nonlinear constitutive equation, the $\mathbf{T}_{\mathbf{E}_1}$ constitutive equation can be used to strains of 10 to 20% in each of the problems presented in Sections 4–7.

In this paper we considered only second-order constitutive equations for the Cauchy stress because of this stress measure's clear physical interpretation as the stress in the deformed body. The response function for other measures of stress can similarly be expanded to second order; and these expressions yield different second-order constitutive equations (Ogden, 1984, p. 350). We obtained the second-order constitutive equations in the displacement gradient, the Biot strain, and the Green strain for both the first Piola–Kirchhoff stress and the second Piola–Kirchhoff stress. The boundary value problems presented in Sections 4–7 were solved with these constitutive equations. The results showed that the choice of the stress measure does result in larger percent differences in the solutions for some of the problems and smaller percent differences in the solutions of other problems. But in all problems, the constitutive equation which is second order in the Biot strain still produces the most accurate results over the widest range of strains and rotations.

In order to determine whether a specific second-order constitutive equation provides a sufficiently good model of mechanical behavior for a real material in a particular deformation regime, it is, of course, necessary to compare the predictions of that second-order constitutive equation with experimental results. But, in the absence of such experimental information, the examples presented in this paper suggest that for elastic materials with properties similar to the Blatz–Ko and the harmonic materials, the constitutive equation which is second-order in the Biot strain is the most accurate over the largest range of strains and rotations.

Acknowledgements

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References

Abeyaratne, R., Horgan, C.O., 1984. The pressurized hollow sphere problem in finite elastostatics for a class of compressible materials. International Journal of Solids and Structures 20, 715–725.

Beatty, M.F., Stalnaker, D.O., 1986. The Poisson function of finite elasticity. Journal of Applied Mechanics 53, 807–813.

Beatty, M.F., 1987. Topics in finite elasticity: hyperelasticity of rubber, elastomers, and biological tissues — with examples. Applied Mechanics Reviews 40, 1699–1734.

Blatz, P.J., Ko, W.L., 1962. Application of finite elastic theory to the deformation of rubbery materials. Transactions of the Society of Rheology 6, 223–251.

Ciarlet, P.G., 1988. Mathematic Elasticity. Elsevier, Amsterdam.

Gurtin, M., 1984. Introduction to Continuum Mechanics. Academic Press, London.

Haughton, D.M., 1993. Shearing of compressible elastic cylinders. Quarterly Journal of Mechanics and Applied Mathematics 49, 471–486.

Haughton, D.M., Lindsay, K.A., 1993. The second-order deformation of a finite compressible isotropic elastic annulus subjected to circular shearing. Proceedings of the Royal Society of London Series A 442, 621–639.

Haughton, D.M., Lindsay, K.A., 1994. The second-order deformation of a finite incompressible isotropic elastic annulus subjected to circular shearing. Acta Mechanica 104, 125–141.

Hoger, A., 1993. Residual stress in an elastic body: a theory for small strains and arbitrary rotations. Journal of Elasticity 31, 1–24.

Hoger, A., 1998. A second-order constitutive theory for hyperelastic materials. International Journal of Solids and Structures 36, 847–868.

Humphrey, J.D., 1995. Mechanics of the arterial wall: review and directions. Critical Reviews in Biomedical Engineering 23, 1–162.

Jafari, A.H., Abeyaratne, R., Horgan, C.O., 1984. The finite deformation of a pressurized circular tube for a class of compressible materials. Zeitschrift für Angewandte Mathematik und Physik 35, 227–246.

John, F., 1960. Plane strain problems for a perfectly elastic material of harmonic type. Communications on Pure and Applied Mathematics 13, 239–296.

John, F., 1966. Plane elastic waves of finite amplitude: Hadamard materials and harmonic materials. Communications of Pure and Applied Mathematics 19, 309–341.

Kirchhoff, G., 1852. Über die Gleichungen des Gleichgewichts eines elastischen Körpers bei nicht unendlich kleinen Verschiebungen seiner Theile. Sitzgsber. Akad. Wiss. Wien 9, 762–773.

Knowles, J.K., Sternberg, E., 1975a. On the singularity induced by certain mixed boundary conditions in linearized and nonlinear elastostatics. International Journal of Solids and Structures 11, 1173–1201.

Knowles, J.K., Sternberg, E., 1975b. On the ellipticity of the equation of nonlinear electrostatics for a special material. Journal of Elasticity 5, 341–361.

Lindsay, K.A., 1992. The second-order deformation of an incompressible isotropic slab under torsion. Quarterly Journal of Mechanics and Applied Mathematics 45, 529–544.

Malvern, L.E., 1969. Introduction to the Mechanics of a Continuous Medium. Prentice-Hall, Englewood Cliffs, NJ.

Murnaghan, F.D., 1937. Finite deformations of an elastic solid. American Journal of Mathematics 59, 235–260.

Murnaghan, F.D., 1951. Finite Deformation of an Elastic Solid. Dover, New York.

Ogden, R.W., Isherwood, D.A., 1978. Solution of some finite plane strain problems for compressible elastic solids. Quarterly Journal of Mechanics and Applied Mathematics 31, 219–249.

Ogden, R.W., 1984. Non-linear Elastic Deformations. Ellis Horwood, Chichester, Great Britain.

Polignone, D.A., Horgan, C.O., 1992. Axisymmetric finite anti-plane shear of compressible nonlinearly elastic circular tubes. Quarterly of Applied Mathematics 50, 323–341.

Polignone, D.A., Horgan, C.O., 1994. Pure azimuthal shear of compressible nonlinearly elastic circular tubes. Quarterly of Applied Mathematics 52, 113–131.

Rivlin, R.S., 1952. The solution of problems in second-order elasticity theory. Journal of Rational Mechanics and Analysis 2, 53–81.

de St Venant, A-J-C.B., 1844. Sur les pressions qui se développent à l'intérieur des corps solides lorsque les déplacements de leurs points, sans altérer l'élasticité, ne peuvent cependant pas être considérés comme très petits. Bull. Soc. Philomath. 5, 26–28.

Sheng, P-L., 1955. Secondary elasticity. Chinese Association of Advanced Science, Monograph Series 1, I, No. 1.

Toupin, R.A., Bernstein, B., 1961. Sound waves in deformed perfectly elastic materials: acousto-elastic effect. Journal of the Acoustical Society of America 33, 216–225.

Truesdell, C., Toupin, T., 1960. The classical field theories. In: Handbuch der Physik. III/1. Springer-Verlag, Berlin.

Truesdell, C., Noll, W., 1965. The non-linear field theories of mechanics. In: Handbuch der Physik. III/3. Springer-Verlag, Berlin.

Wineman, A.S., Waldron Jr, W.K., 1995. Normal stress effects induced during circular shear of a compressible non-linear elastic cylinder. International Journal of Non-Linear Mechanics 30, 323–339.